

Chapter 3: State Variable Models

From the Chapter 2 Motor to Internal Dynamics and Initial Conditions

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The Chapter 2 model answered only an input–output question

The Chapter 2 motor ODE and transfer function predict shaft angle from voltage:

$$G_{\theta}(s) = \frac{\Theta(s)}{V(s)} = \frac{2}{s(s^2 + 12s + 20.02)}.$$

The missing information

A transfer function assumes zero initial conditions and hides current and angular speed. It cannot by itself answer, “what happens from the present internal condition?”

voltage $v \longrightarrow$ **motor** $\longrightarrow$ angle θ
speed ω , current i , stored energy, and
initial state $x(0)$

State variables open the box without
abandoning the verified input–output model.

Why state variables are the next modelling language

Transfer-function view

external map, compact algebra, zero-state response

State-space view

internal memory, multiple outputs, initial state, simulation-ready equations

This chapter uses one plant and one state order throughout:

$$\mathbf{x} = [\theta \ \omega \ i]^T$$

Here θ is angle [rad], $\omega = \dot{\theta}$ is speed [rad/s], and i is armature current [A]. Superscript T transposes a row into a column. Only an explicitly taught similarity transformation will change coordinates.

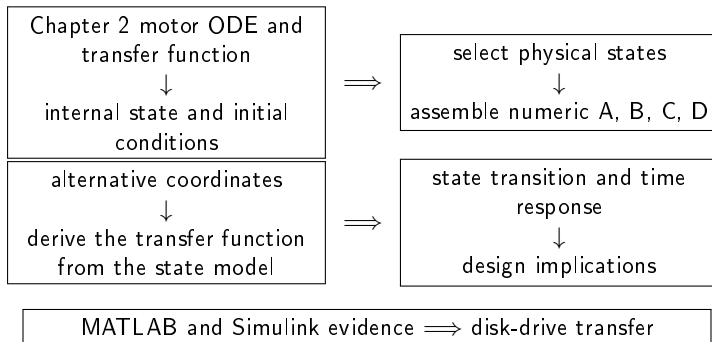
Design check

Every state must have a physical or mathematical reason to remember the past.

Learning route: five conceptual phases

Learning outcomes

- select physical states, assemble A, B, C, D , and trace connected realizations;
- transform coordinates, recover $G(s)$, and compute modes and state response;
- verify the model in software and transfer the method to a flexible disk drive.



3.1 Introduction: from external map to internal memory

Link from Chapter 2

The ODE and G_θ agree for zero initial conditions. The next question is: what minimum information at time t_0 makes every future response predictable?

For a linear time-invariant system,

$$\boxed{\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}}, \quad \boxed{\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}}.$$

$\mathbf{x}$ stores internal memory; $\mathbf{u}$ supplies external causes; $\mathbf{y}$ selects what is measured. $\mathbf{A}$ maps state to state rate; $\mathbf{B}$ maps input to state rate; $\mathbf{C}$ selects outputs; $\mathbf{D}$ represents direct input-to-output transmission.

Design check

State-space adds initial-condition and internal-variable prediction while retaining the transfer-function result as a testable consequence.

What the transfer function deliberately suppresses

- $G(s)$ maps input to output under zero initial conditions.
- It does not expose $i(t)$ or $\omega(t)$ unless they are chosen outputs.
- It cannot represent the independent response generated by $\mathbf{x}(0)$.

Takeaway

To remove ambiguity, choose a minimal set of variables that captures every independent energy store and kinematic accumulator.

external equivalence $\nRightarrow$ identical internal coordinates

Next: which motor variables are sufficient, and which are redundant?

Recall the physical motor parameters

J	rotor inertia	0.01 kg m^2
b	viscous friction coefficient	0.10 N m s/rad
K_t	torque per current	0.01 N m/A
K_e	back-emf per angular speed	0.01 V s/rad
R	winding resistance	1Ω
L_a	winding inductance	0.5 H

Torque is $\tau_m = K_t i$ and back-emf is $K_e \omega$. The independent physical initial values are angle, speed, and current.

A minimal input-output realization additionally requires controllability and observability, which we test in Section 3.8.

3.2 State means sufficient memory, not every variable

Definition

The state at t_0 is the minimum information that, together with future input and the model, determines the future system trajectory.

Independent memory

winding flux/current i , rotational momentum $J\omega$, and accumulated angle θ .

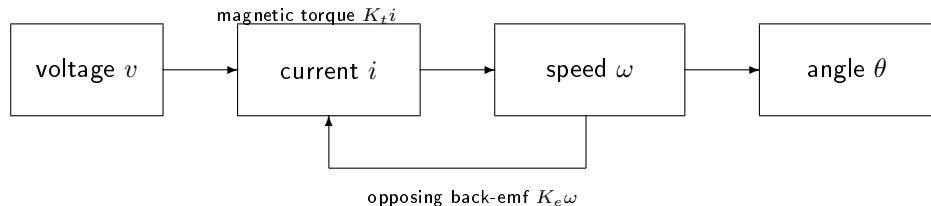
Derived variables

torque $K_t i$, back-emf $K_e \omega$, and measured angle $y = \theta$ follow algebraically.

Minimality test

Removing i , ω , or θ loses an independent initial condition; adding motor torque creates redundancy because $\tau_m = K_t i$.

Physical states follow the motor's energy and kinematic chain



State	memory mechanism	unit
θ	kinematic integral of speed	rad
ω	rotational momentum $J\omega$	rad/s
i	winding magnetic energy $L_a i^2/2$	A

Design check

The arrow topology anticipates the nonzero entries of A before arithmetic begins.

Interaction: Select a minimal motor state set

Predict – Which candidate is a minimal state for the motor when output is angle?

1. $[\theta \ \omega \ i]^T$
2. $[\theta \ \omega \ i \ K_t i]^T$
3. $[\theta \ K_t i]^T$

Hint

Count independent integrators and energy stores; algebraic copies add no memory.

Interaction: Select a minimal motor state set

Predict – Which candidate is a minimal state for the motor when output is angle?

1. $[\theta \ \omega \ i]^T$
2. $[\theta \ \omega \ i \ K_t i]^T$
3. $[\theta \ K_t i]^T$

Answer

Choice 1. Choice 2 is redundant because torque is algebraic; choice 3 loses rotational momentum.

Two matrix operations used in the derivation

I_n is the $n \times n$ identity matrix and $(sI_n - A)^{-1}$ is the inverse of $sI_n - A$ wherever it exists. The determinant vanishes at eigenvalues of A :

$$Av = \lambda v, \quad v \neq 0 \quad \Longleftrightarrow \quad \det(\lambda I_n - A) = 0.$$

Each eigenvalue describes a natural mode; similarity transformations change its coordinates, not its value. In this motor $n = 3$. Capital signal letters denote Laplace transforms, not new physical variables.

State coordinates are not unique; the physical order is our anchor

If T is nonsingular, $z = T\mathbf{x}$ stores the same information in different coordinates. Yet coordinates affect interpretation.

$$\mathbf{x} = [\theta \ \omega \ i]^T \quad \Longrightarrow \quad \begin{array}{l} \text{direct units and energy meaning} \\ \text{transparent link to Chapter 2 equations} \end{array}$$

Takeaway

We keep the physical order fixed until the later coordinate-change step explicitly teaches a change. Next we assemble numeric A, B, C, D from the same equations.

3.3 Each physical equation becomes one row

With $\tau_L = 0$ and the Chapter 2 sign convention,

$$\begin{aligned}\dot{\theta} &= \omega, \\ J\dot{\omega} &= -b\omega + K_t i, \\ L_a \dot{i} &= -K_e \omega - Ri + v.\end{aligned}$$

Divide each dynamic balance by its storage parameter:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -b/J & K_t/J \\ 0 & -K_e/L_a & -R/L_a \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1/L_a \end{bmatrix} v.$$

The zero first column means absolute angle does not feed back into this open-loop plant.

Worked calculation 1: predict the instantaneous state direction

Use the preceding motor equations and the Chapter 2 SI parameter values.

Given: $\mathbf{x} = [0.10 \text{ rad}, 2.00 \text{ rad/s}, 1.50 \text{ A}]^T$, $v = 6.00 \text{ V}$.

Model: $\dot{\theta} = \omega$, $\dot{\omega} = -10\omega + i$, $\dot{i} = -0.02\omega - 2i + 2v$.

Substitution:

$$\dot{\theta} = 2.00, \quad \dot{\omega} = -10(2.00) + 1.50 = -18.50, \quad \dot{i} = -0.02(2.00) - 2(1.50) + 2(6.00) = 8.96.$$

Result:

$$\dot{\mathbf{x}} = [2.00 \text{ rad/s}, -18.50 \text{ rad/s}^2, 8.96 \text{ A/s}]^T.$$

Check: positive voltage raises current, while viscous torque dominates the initial acceleration; both signs are physically plausible.

Worked calculation 2: numeric motor matrices

The three physical balance rows now become the numeric model used by every later realization and software check.

Given: $J = 0.01 \text{ kg m}^2$, $b = 0.10 \text{ N m s/rad}$, $K_t = 0.01 \text{ N m/A}$, $K_e = 0.01 \text{ V s/rad}$, $R = 1 \Omega$, $L_a = 0.50 \text{ H}$; v in V and $y = \theta$ in rad.

Model: $A_{22} = -b/J$, $A_{23} = K_t/J$, $A_{32} = -K_e/L_a$, $A_{33} = -R/L_a$, $B_3 = 1/L_a$.

Substitution:

$$A_{22} = -0.10/0.01 = -10 \text{ s}^{-1},$$

$$A_{23} = 0.01/0.01 = 1 \text{ rad/(s}^2 \text{ A)},$$

$$A_{32} = -0.01/0.50 = -0.02 \text{ A/rad},$$

$$A_{33} = -1/0.50 = -2 \text{ s}^{-1}, \quad B_3 = 2 \text{ A/(V s)}.$$

Result:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -10 & 1 \\ 0 & -0.02 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \quad C = [1 \quad 0 \quad 0], \quad D = 0.$$

Check: multiplying each row by $[\text{rad}, \text{rad/s}, \text{A}]^T$ returns $[\text{rad/s}, \text{rad/s}^2, \text{A/s}]^T$; dimensions are consistent.

Interaction: Assemble one row of the motor A matrix

Calculate – For $L_a = 0.50$, $R = 1$, and $K_e = 0.01$, which row maps $[\theta \ \omega \ i]^T$ into $\dot{i}$?

1. $[0 \ -2 \ -0.02]$
2. $[0 \ -0.02 \ -2]$
3. $[-0.02 \ 0 \ -2]$

Hint

Write $\dot{i} = (v - Ri - K_e\omega)/L_a$ before inserting numbers.

Interaction: Assemble one row of the motor A matrix

Calculate – For $L_a = 0.50$, $R = 1$, and $K_e = 0.01$, which row maps $[\theta \ \omega \ i]^T$ into $\dot{i}$?

1. $[0 \ -2 \ -0.02]$
2. $[0 \ -0.02 \ -2]$
3. $[-0.02 \ 0 \ -2]$

Answer

Choice 2: angle has no direct effect; speed contributes $-K_e/L_a = -0.02$ and current contributes $-R/L_a = -2$.

Dimensions and output choice are part of the model

For $n = 3$ states, $m = 1$ input, and $p = 1$ measured output,

$$A \in \mathbb{R}^{3 \times 3}, \quad B \in \mathbb{R}^{3 \times 1}, \quad C \in \mathbb{R}^{1 \times 3}, \quad D \in \mathbb{R}^{1 \times 1}.$$

Angle output

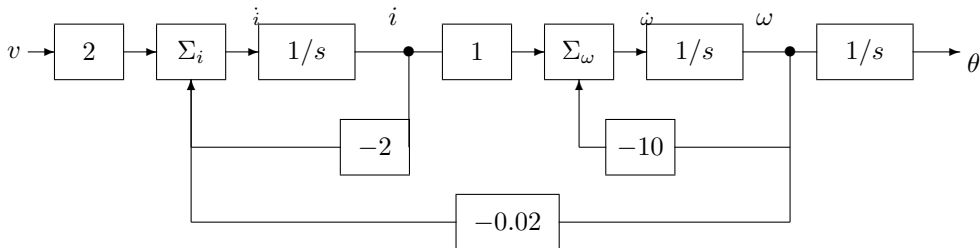
$$C_\theta = [1 \ 0 \ 0], \quad D = 0$$

Changing C changes the reported output, not the internal motor dynamics A, B .

All states for evidence

$$C_x = I_3, \quad D_x = 0_{3 \times 1}$$

The numeric matrix is an executable causal map

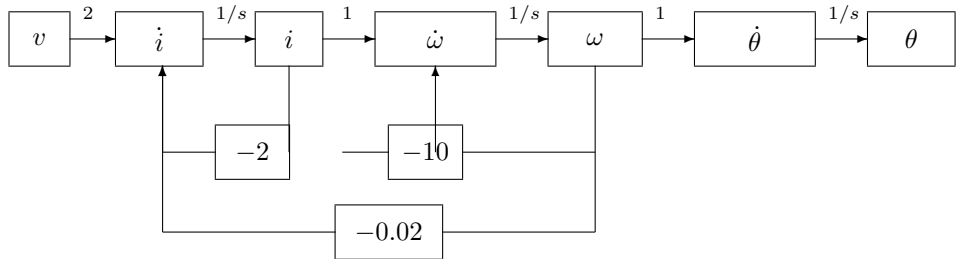


The unique sums implement $\dot{i} = 2v - 2i - 0.02\omega$ and $\dot{\omega} = i - 10\omega$; every return branch visibly rejoins its destination.

Takeaway

Rows become summing junctions; states are integrator outputs; coefficients become signed gains. Next we compare standard realization layouts without changing behavior.

3.4 A state-flow graph is the matrix written as paths

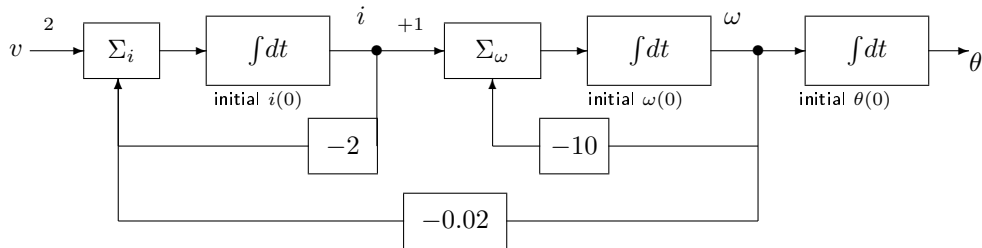


A branch $x_j \rightarrow \dot{x}_k$ carries A_{kj} ; an input branch $u \rightarrow \dot{x}_k$ carries B_{kj} . Exact signs remain live in LaTeX.

Reading rule

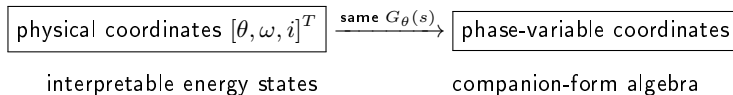
Trace the destination derivative first; this prevents row/column transposition errors.

Block realization: derivatives enter, states leave integrators



One integrator stores each state; its initial value is the matching component of $\mathbf{x}(0)$. The connected returns reproduce every nonzero entry of $\mathbf{A}$.

The same input–output map admits different realizations



Takeaway

A diagram is a realization, not a unique physical truth. To compare realizations safely, we need a nonsingular coordinate map and its transformed matrices.

Next: transform coordinates without changing external behavior.

3.5 Alternative coordinates require a nonsingular map

Define $z = Tx$ with nonsingular T . Then $x = T^{-1}z$ and

$$\dot{z} = \underbrace{TAT^{-1}}_{A_z} z + \underbrace{TB}_{B_z} v, \quad y = \underbrace{CT^{-1}}_{C_z} z + Dv.$$

This is a **similarity transformation**: it changes coordinates, not physical trajectories or the input–output map.

Definition before use

T must be invertible; otherwise two distinct physical states could collapse into one coordinate and information would be lost.

Worked 3: reorder motor states

This deliberately simple coordinate change isolates the transformation rule and then reconnects to the same motor.

Given: $\mathbf{x} = [\theta \ \omega \ i]^T$ and $\mathbf{z} = [i \ \theta \ \omega]^T = T\mathbf{x}$, with $T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Model: $A_z = TAT^{-1}$ and $B_z = TB$.

Substitution:

$$A_z = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} A \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -0.02 \\ 0 & 0 & 1 \\ 1 & 0 & -10 \end{bmatrix}, \quad B_z = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.$$

Result: $A_z = \begin{bmatrix} -2 & 0 & -0.02 \\ 0 & 0 & 1 \\ 1 & 0 & -10 \end{bmatrix}$, $B_z = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ and $\dot{\mathbf{z}} = A_z\mathbf{z} + B_zv$.

Check: $\mathbf{z}$ has units $[A, \text{rad}, \text{rad/s}]^T$ and $\dot{\mathbf{z}}$ has $[A/s, \text{rad/s}, \text{rad/s}^2]^T$; the first row remains $\dot{i} = -2i - 0.02\omega + 2v$.

What similarity preserves—and what it changes

Preserved

eigenvalues, characteristic polynomial, transfer function, and physical output trajectory

Changed

state labels, individual matrix entries, numerical conditioning, and direct physical interpretation

$$\det(sI - TAT^{-1}) = \det(T(sI - A)T^{-1}) = \det(sI - A).$$

Takeaway

External equivalence should now be derivable directly from A, B, C, D . Next we recover $G_\theta(s)$ and compare it with Chapter 2.

3.6 Eliminate the state only after declaring zero initial conditions

Take the Laplace transform of $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$:

$$s\mathbf{X}(s) - \mathbf{x}(0) = A\mathbf{X}(s) + B\mathbf{U}(s).$$

When $\mathbf{x}(0) = 0$,

$$\mathbf{X}(s) = (sI - A)^{-1}B\mathbf{U}(s),$$

and therefore

$$\boxed{G(s) = C(sI - A)^{-1}B + D}.$$

Contract notation: $G = C(sI - A)^{-1}B + D$.

Scope

This equality is an input-output statement for zero initial conditions; it is not a claim that the transfer function contains the initial-state response.

Worked calculation 4: recover $G_\theta(s)$ from the state model

This calculation closes the loop back to the Chapter 2 transfer function and licenses the later equivalence test.

Given: v in V, θ in rad, $\mathbf{x}(0) = 0$, and $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -10 & 1 \\ 0 & -0.02 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$, $C_\theta = [1 \quad 0 \quad 0]$, $D = 0 \text{ rad/V}$.

Model: $G_\theta = C_\theta(sI - A)^{-1}B$.

Substitution:

$$sI - A = \begin{bmatrix} s & -1 & 0 \\ 0 & s+10 & -1 \\ 0 & 0.02 & s+2 \end{bmatrix}, \quad \det(sI - A) = s[(s+10)(s+2) + 0.02] = s(s^2 + 12s + 20.02).$$

Write $I_a(s)$ for transformed current, distinct from identity matrix I . The three rows give $\Omega = s\Theta$, $I_a = (s+10)\Omega$, and $(s+2)I_a + 0.02\Omega = 2V$. Thus $[s^2 + 12s + 20.02]s\Theta = 2V$.

Result:

$$G_\theta(s) = \frac{2}{s(s^2 + 12s + 20.02)} \text{ rad/V} = \frac{0.01}{0.005s^3 + 0.06s^2 + 0.1001s}.$$

Check: numerator and denominator match Chapter 2 after dividing by 0.005; order three matches three independent states.

Software equivalence has a precise boundary

A numerical comparison should test two different questions: the first under zero initial conditions and the second under nonzero initial conditions.

1. **Zero initial conditions:** state-space and transfer-function outputs must agree.
2. **Nonzero initial conditions:** state-space predicts a free response that a zero-IC transfer simulation does not contain.

Equivalence claim

same $v(t)$, same zero state, same $\theta(t)$ within normalized error $< 1\%$

Next: write the complete response that contains both sources.

Distinction claim

with $v = 0$ and $x(0) \neq 0$, the initial state is a separate response source

3.7 The complete response separates memory from forcing

Define the **state transition matrix**

$$\Phi(t) = e^{At} = I + At + \frac{A^2 t^2}{2!} + \dots$$

Then, for $t \geq 0$,

$$\mathbf{x}(t) = \underbrace{e^{At} \mathbf{x}(0)}_{\text{zero-input response}} + \underbrace{\int_0^t e^{A(t-\tau)} B \mathbf{u}(\tau) d\tau}_{\text{zero-state response}}.$$

Design check

$\Phi(0) = I$ and $\frac{d}{dt}\Phi(t) = A\Phi(t)$ are quick algebraic checks.

Modes are the exponential components associated with eigenvalues of A .

Worked 5: interpret motor modes

The poles recovered in Section 3.6 now become internal modal rates for response and design.

Given: $\det(\lambda I - A) = \lambda[\lambda^2 + (12\text{ s}^{-1})\lambda + 20.02\text{ s}^{-2}] = 0$.

Model: eigenvalues solve $\det(\lambda I - A) = 0$; a stable real mode has $\tau = -1/\lambda$.

Substitution:

$$\lambda_{2,3} = \frac{-12\text{ s}^{-1} \pm \sqrt{(12\text{ s}^{-1})^2 - 4(20.02\text{ s}^{-2})}}{2} = \frac{-12 \pm 7.995}{2} \text{ s}^{-1} \Rightarrow -2.0025, -9.9975 \text{ s}^{-1}.$$

Thus $\tau_{\text{slow}} = 1/2.0025 = 0.4994\text{ s}$ and $\tau_{\text{fast}} = 1/9.9975 = 0.1000\text{ s}$.

Result:

$$\lambda(A) = \{0, -2.0025, -9.9975\} \text{ s}^{-1}, \quad \tau = \{\infty, 0.4994, 0.1000\} \text{ s}.$$

Check: the pole at zero is the angle integrator; the two negative modes decay, with electrical/mechanical coupling perturbing the nominal -2 and -10 rates only slightly.

Interaction: Interpret an eigenvalue as a mode

Interpret – Which statement best describes $\lambda = -2.0025 \text{ s}^{-1}$?

1. It is an unstable oscillation at 2.0025 rad/s .
2. It is a decaying mode with time constant about 0.50 s .
3. It is the angle integrator.

Hint

For a real stable mode $e^{\lambda t}$, use $\tau = -1/\lambda$.

Interaction: Interpret an eigenvalue as a mode

Interpret – Which statement best describes $\lambda = -2.0025 \text{ s}^{-1}$?

1. It is an unstable oscillation at 2.0025 rad/s .
2. It is a decaying mode with time constant about 0.50 s .
3. It is the angle integrator.

Answer

Choice 2. The sign gives decay and $1/2.0025 \approx 0.50 \text{ s}$; the integrator is the zero eigenvalue.

Worked calculation 6: propagate motor memory

Removing only the pure angle integrator isolates the $[\omega \ i]^T$ memory calculation, then reconnects it to the full motor.

Given: $A_m = \begin{bmatrix} -10 & 1 \\ -0.02 & -2 \end{bmatrix}$, $[\omega(0), i(0)]^T = [1 \text{ rad/s}, 0 \text{ A}]^T$, $v = 0$, $t = 0.2 \text{ s}$.

Model: $\mathbf{x}_m(t) = \Phi_m(t)\mathbf{x}_m(0)$, $\Phi_m = e^{A_m t}$.

Substitution:

$$\Phi_m(0.2) \approx \begin{bmatrix} 0.1352 & 0.06686 \\ -0.001337 & 0.6702 \end{bmatrix}, \quad \mathbf{x}_m(0.2) = \Phi_m(0.2) \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Result:

$$\omega(0.2) \approx 0.1352 \text{ rad/s}, \quad i(0.2) \approx -1.337 \text{ mA}.$$

Check: speed decays below its initial value; the small negative current is the back-emf response. In the full model, θ accumulates this decaying speed. These modes now raise two design questions: can voltage move every state, and can angle reveal every state?

3.8 State models expose two design questions

Controllability

Can the input move every state direction? Test $\mathcal{C} = [B \ AB \ A^2B]$.

Observability

Can the chosen output distinguish every state direction? Test $\mathcal{O} = [C^T \ (CA)^T \ (CA^2)^T]^T$.

Design check

For this three-state model, full rank means rank 3. Rank loss proves a direction cannot be controlled or reconstructed with the chosen channels.

These properties connect modelling choices directly to controller and observer design. In short, controllability asks about actuation reach and observability asks about sensing reach.

Why successive powers of A appear

An input first acts in direction B ; dynamics propagate it through $AB, A^2B, \dots$. Their span is the controllable part of state space. With zero input, successive output derivatives satisfy

$$y = Cx, \quad \dot{y} = CAx, \quad \ddot{y} = CA^2x.$$

These rows describe which state directions can be distinguished from output. For an n -state system, collect powers through A^{n-1} . Rank n means all state directions are accessible or distinguishable, respectively. The next motor calculation uses $n = 3$.

Worked 7: angle observability

This rank calculation tests whether the angle-only output chosen in Section 3.3 preserves enough dynamic information for later estimation.

Given: $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -10 & 1 \\ 0 & -0.02 & -2 \end{bmatrix}$ with row derivative units $[\text{rad/s}, \text{rad/s}^2, \text{A/s}]^T$, and $C_\theta = [1 \quad 0 \quad 0]$ for angle in rad.

Model: $\mathcal{O} = \begin{bmatrix} C_\theta \\ C_\theta A \\ C_\theta A^2 \end{bmatrix}$; full rank 3 means observable.

Substitution:

$$C_\theta A = [0 \ 1 \ 0], \quad C_\theta A^2 = [0 \ -10 \ 1], \quad \mathcal{O} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -10 & 1 \end{bmatrix}.$$

In the displayed SI numerical coordinates, row reduction leaves I_3 . Rank is unchanged by nonzero state-unit rescaling.

Result: $\text{rank}(\mathcal{O}) = 3$ (dimensionless): observable.

Check: differentiating angle exposes speed; its next dynamic relation contains current. The rank tests are analytic; next MATLAB and Simulink provide independent executable evidence for trajectories and initial-condition scope.

3.9 MATLAB builds one state model from the equations

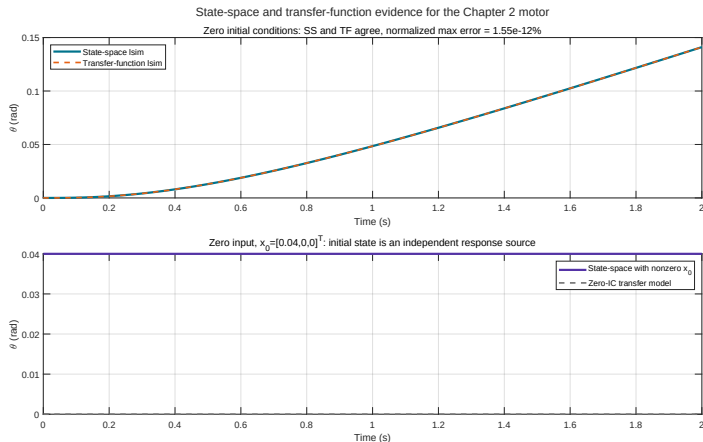
Compare the same 1-V step and zero initial state in both representations.

```
A = [0 1 0; 0 -10 1; 0 -0.02 -2];  
B = [0;0;2]; Ctheta = [1 0 0]; D = 0;  
sysState = ss(A,B,Ctheta,D);  
Gtheta = tf(0.01,[0.005 0.06 0.1001 0]);  
t = (0:0.001:8)'; u = ones(size(t));  
[ySS,~,x] = lsim(sysState,u,t,zeros(3,1));  
yTF = lsim(Gtheta,u,t);  
normalizedMaxError = max(abs(ySS-yTF))/max(1,max(abs(yTF)));  
assert(normalizedMaxError < 0.01)
```

Design check

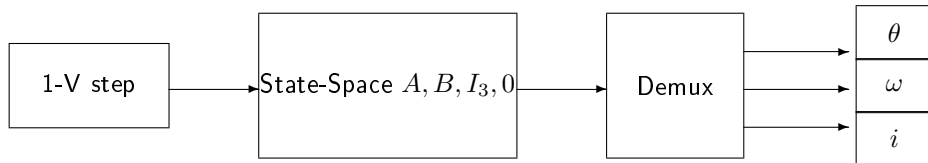
The script also asserts sizes, exact numeric entries, transfer coefficients, eigenvalues, and the nonzero-initial-condition distinction.

State/TF evidence: agreement and a deliberate distinction



The top comparison uses identical voltage and zero state; the bottom comparison holds input at zero and changes only $x(0)$.

Headless Simulink model logs every physical state



Simulation check path

The builder creates and saves **ch03_state_space_servo.slx** without opening an editor. Signal logging names are θ , ω , and current; each is interpolated onto the MATLAB reference grid and compared with the corresponding *lsim* state.

Executable acceptance is componentwise, not just visual

Evidence	comparison	gate
matrices	numeric A, B, C, D versus parameter-derived entries	exact to tolerance
zero-state output	state-space $\theta(t)$ versus $G_\theta(s)$	normalized max error $< 1\%$
initial state	$\mathbf{x}(0) = [0.04, 0, 0]^T$ versus zero-IC TF	visible nonzero gap
Simulink states	logged θ, ω, i versus MATLAB <i>lsim</i>	each error $< 1\%$

Failure diagnosis order

Check state order, C output definition, initial conditions, then parameters and solver settings.

Interaction: Diagnose a state–transfer mismatch

Diagnose – MATLAB state-space starts at $\theta(0) = 0.04\text{ rad}$ with $v = 0$, but a transfer-function simulation starts at zero. What is the primary cause?

1. The state order is invalid.
2. The transfer model represents the zero-initial-condition input–output map.
3. The motor must be unobservable.

Hint

Ask which response source is present in only one experiment.

With state order, output, and initial conditions aligned, the same discipline now transfers to a flexible disk-drive model.

Interaction: Diagnose a state–transfer mismatch

Diagnose – MATLAB state-space starts at $\theta(0) = 0.04 \text{ rad}$ with $v = 0$, but a transfer-function simulation starts at zero. What is the primary cause?

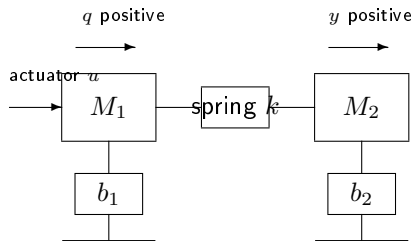
1. The state order is invalid.
2. The transfer model represents the zero-initial-condition input–output map.
3. The motor must be unobservable.

Answer

Choice 2. The state experiment includes $e^{At}\mathbf{x}(0)$; the zero-IC transfer experiment does not. Match initial conditions before blaming coefficients.

With state order, output, and initial conditions aligned, the same discipline now transfers to a flexible disk-drive model.

3.10 Disk drive: flexure adds states, not a cosmetic correction



$$M_1 \ddot{q} = u - b_1 \dot{q} - k(q - y),$$

$$M_2 \ddot{y} = k(q - y) - b_2 \dot{y}.$$

The spring force acts equally and oppositely; both dampers return velocity-dependent force to ground.

Textbook case: $M_1 = 0.020$ kg, $M_2 = 0.0005$ kg, $k = 10$ N/m, $b_1 = 0.410$ N s/m, $b_2 = 0.0041$ N s/m. Thus

$$\frac{k}{M_1} = 500, \quad \frac{k}{M_2} = 20000, \quad \frac{b_1}{M_1} = 20.5, \quad \frac{b_2}{M_2} = 8.2.$$

For $\mathbf{x}_d = [q \ y \ \dot{q} \ \dot{y}]^T$ and force input u in N,

$$\dot{\mathbf{x}}_d = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -500 & 500 & -20.5 & 0 \\ 20000 & -20000 & 0 & -8.2 \end{bmatrix} \mathbf{x}_d + \begin{bmatrix} 0 \\ 0 \\ 50 \\ 0 \end{bmatrix} u.$$

Check: at $k = 10$ N/m the flexure mode remains oscillatory; the textbook rigid-design implication is $k > 100$ N/m, so a one-mass approximation can hide the head motion.

Chapter 3 closes the external–internal evidence loop

1. Chapter 2 exposed a limitation: hidden states and initial conditions.
2. Physical memory selected $\mathbf{x} = [\theta \ \omega \ i]^T$ and produced verified A, B, C, D .
3. Live realizations and a similarity transformation separated coordinates from behavior.
4. $G = C(sI - A)^{-1}B + D$ recovered the zero-state transfer model.
5. $e^{At}\mathbf{x}(0)$ restored the missing initial-state response.
6. MATLAB and Simulink independently checked output and all three states.
7. The disk drive showed how a flexure adds internal modes that a rigid approximation hides.

Takeaway to Chapter 4

A trustworthy state model now supplies poles, modes, and complete time response—the quantities needed to judge dynamic performance and design feedback.

Homework

E3.2, P3.12, P3.17

Submit homework to homeworkscut2026@126.com; **due September 24, 2026.**