

# Chapter 4 — Feedback Control System Characteristics

Dorf & Bishop, Modern Control Systems (14th ed.)

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# Learning Outcomes

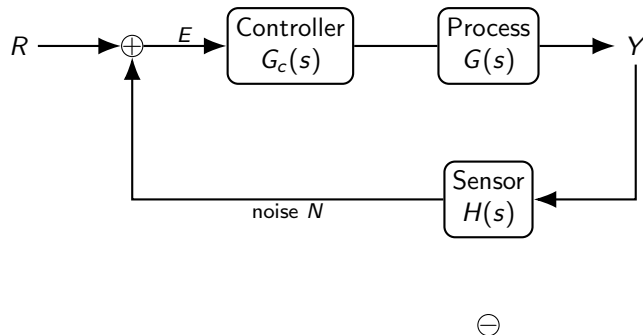
By the end of this tutorial you will be able to:

- 1 Explain the central role of the **error signal** in closed-loop control.
- 2 Derive and interpret **loop gain**  $L(s)$ , **sensitivity**  $S(s)$ , and **complementary sensitivity**  $C(s)$ .
- 3 Analyze **parameter sensitivity** using normalized sensitivities.
- 4 Evaluate **disturbance rejection** and **measurement noise attenuation** from  $E(s)$  and  $Y(s)$  relations.
- 5 Shape **transient response** (poles, time constants) with feedback.
- 6 Compute **steady-state error** via the final value theorem.
- 7 Weigh the **costs of feedback**: complexity, gain loss, potential instability.
- 8 Apply the above to **two design examples**: boring machine and blood pressure control; and review a **sequential design** (disk drive read system).

## 4.1 Introduction: Open vs. Closed Loop

Open-loop:  $Y(s) = G(s) R(s)$ ; no correction; sensitive to  $G$  changes and disturbances.

Closed-loop: measure  $Y$ , compare with  $R$ , drive **error**  $E = R - Y$  through controller  $G_c$ .



# Closed-Loop Advantages

- Reduced **sensitivity** to process parameter changes.
- Improved **disturbance rejection**.
- Improved **measurement noise attenuation** (when shaped appropriately).
- **Lower steady-state error** and tunable **transient response**.

## 4.2 Error Signal Analysis: Definitions (with intuition)

**Setup (unity feedback  $H = 1$ ).** We use

$$E(s) = R(s) - Y(s), \quad L(s) = G_c(s)G(s), \quad S(s) = \frac{1}{1 + L(s)}, \quad C(s) = \frac{L(s)}{1 + L(s)}.$$

**What each symbol means (plain English):**

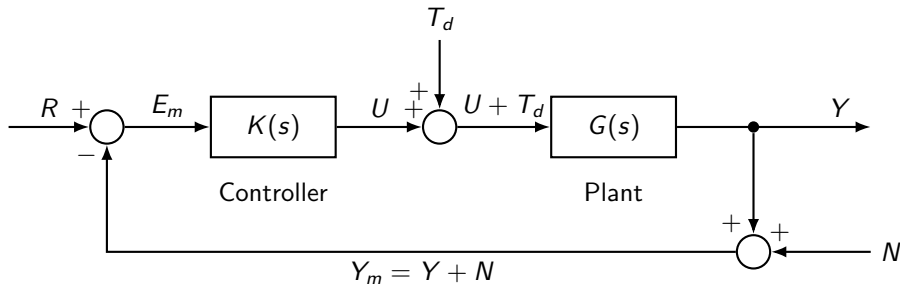
- $L$  (loop gain) = “how hard the loop pushes back.” Big  $|L|$  means strong correction.
- $S$  (sensitivity) = “how much gets through despite feedback.” Small  $|S|$  is good for tracking & disturbance rejection.
- $C$  (complementary sensitivity) = “how much the loop passes what it measures.” Small  $|C|$  is good for noise attenuation.

### Key trade-off

$S + C = 1$  at each frequency  $\Rightarrow$  you can't make both small at the *same* frequency. Design idea: make  $|L|$  large at low  $\omega$  (kill disturbances) and small at high  $\omega$  (avoid injecting sensor noise).

# Tracking error: define the signals and the loop

**Input locations matter:**  $T_d$  is added before the plant;  $N$  is added only to the measurement.



## Physical tracking error

$$E = R - Y.$$

The difference between the reference and the true output.

## Error received by the controller

$$E_m = R - Y_m = R - Y - N = E - N.$$

The controller acts on  $E_m$ , not directly on  $E$ .

# Derive the error equation from the block diagram

SISO LTI model, zero initial conditions; the argument  $s$  is omitted below.

Read the signal relations from the diagram

$$E = R - Y, \quad E_m = E - N, \quad U = KE_m, \quad Y = G(U + T_d).$$

Substitute along the physical signal path:

$$E = R - Y$$

definition of the physical error

$$= R - G(U + T_d)$$

plant equation

$$= R - G(KE_m + T_d)$$

controller equation

$$= R - GK(E - N) - GT_d$$

substitute  $E_m = E - N$

$$= R - GKE + GKN - GT_d$$

expand and keep the noise sign.

Move the term containing  $E$  to the left:

$$(1 + GK)E = R - GT_d + GKN.$$

# Recognize sensitivity and complementary sensitivity

**Define the loop transfer function and the two sensitivities:**

$$L = GK, \quad S = \frac{1}{1+L}, \quad C = \frac{L}{1+L}, \quad S + C = 1.$$

Here  $C$  denotes *complementary sensitivity*, not the controller.

$$E = \frac{1}{1+GK}R - \frac{G}{1+GK}T_d + \frac{GK}{1+GK}N.$$

**Physical tracking error:  $E = R - Y$**

$$E(s) = \underbrace{S(s)R(s)}_{\text{reference contribution}} - \underbrace{S(s)G(s)T_d(s)}_{\text{plant-input disturbance}} + \underbrace{C(s)N(s)}_{\text{sensor-noise contribution}}.$$

**Superposition: consider one input at a time.**

$$E_R = SR, \quad E_d = -SGT_d, \quad E_N = CN, \quad E = E_R + E_d + E_N.$$

# Why is the measurement-noise term positive?

Check the output equation independently:

$$\begin{aligned}Y &= GK(R - Y - N) + GT_d, \\(1 + GK)Y &= GKR + GT_d - GKN, \\Y &= CR + SGT_d - CN.\end{aligned}$$

## Physical tracking error

$$E = R - Y = SR - SGT_d + CN.$$

The noise contribution is  $+CN$  because the noise contribution to  $Y$  is  $-CN$ .

## Measured / comparator error

$$\begin{aligned}E_m &= E - N \\&= SR - SGT_d - SN,\end{aligned}$$

using  $C - 1 = -S$ .

**Noise alone** ( $R = T_d = 0$ ):

$$Y = -CN, \quad E = CN, \quad E_m = -SN.$$

**Key distinction:** a small  $E_m$  does not guarantee a small  $E$ .

# Loop shaping: choose the right frequency bands

Evaluate all transfer functions at  $s = j\omega$ .

Tracking band:  $|L| \gg 1$

$$S \approx \frac{1}{L}, \quad C \approx 1, \quad SG \approx \frac{1}{K}.$$

$$E_R \approx \frac{R}{L}, \quad E_d \approx -\frac{T_d}{K}.$$

Large loop gain reduces reference error and attenuates the disturbance relative to its open-loop effect.

**Limitation:**  $E_N \approx N$ ; sensor noise passes through.

Sensor-noise band:  $|L| \ll 1$

$$S \approx 1, \quad C \approx L.$$

$$E_N = CN \approx LN.$$

Small loop gain reduces the effect of measurement noise on the physical output and physical tracking error.

**Limitation:** reference variations are poorly tracked in this band:  $E_R \approx R$ .

**Disturbance check:** design for  $|SG|$ , not  $|S|$  alone; the magnitude and frequency dependence of  $G$  still matter.

# Checks that prevent a misleading design rule

At a given frequency,  $S$  and  $C$  cannot both be very small

$$S + C = 1 \implies |S(j\omega)| + |C(j\omega)| \geq 1.$$

Tracking and sensor-noise rejection therefore need compatible frequency bands.

Loop gain magnitude alone does not establish a good design

$$|S(j\omega)| = \frac{1}{|1 + L(j\omega)|}.$$

Near  $L = -1$ , sensitivity can become large. Preserve internal closed-loop stability and inspect both gain and phase near crossover.

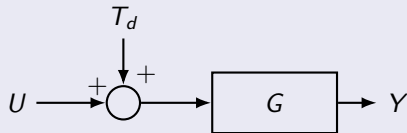
For example, at a frequency where  $L(j\omega_0) = -0.95$ ,

$$S(j\omega_0) = 20, \quad C(j\omega_0) = -19.$$

Both effects can be amplified near  $-1$ ; their complex-valued sum is still one.

# Disturbance modeling: location and sign matter

## Disturbance at the plant input

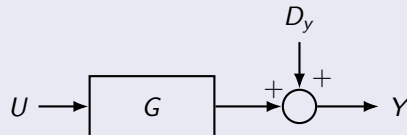


$$Y = G(U + T_d).$$

$$E = SR - SGT_d + CN.$$

It passes through  $G$ ; the error map is  $-SG$ .

## Disturbance at the plant output



$$Y = GU + D_y.$$

$$E = SR - SD_y + CN.$$

It bypasses  $G$ ; the error map is  $-S$ .

## A resisting load torque may use the opposite sign

$$Y = G(U - T_d) \implies E = SR + SGT_d + CN.$$

Use the disturbance sign from the physical model, not from the symbol  $T_d$  alone.

# Tracking Error in Terms of $S$ and $C$ — what each term does

With three inputs (reference  $R$ , plant-input disturbance  $T_d$ , sensor noise  $N$ ):

$$E(s) = \underbrace{S(s) R(s)}_{\text{tracking term}} - \underbrace{S(s) G(s) T_d(s)}_{\text{disturbance term}} + \underbrace{C(s) N(s)}_{\text{noise term}}.$$

**How to read this:**

- $S R$ : where  $|S|$  is small, the loop forces  $Y \approx R$  (small error).
- $-S G T_d$ : where  $|S|$  is small, the plant-input disturbance is suppressed.
- $C N$ : where  $|C|$  is small, sensor noise does not show up as error.

**Tiny numeric example (intuition):**  $G_c = 10$ ,  $G = \frac{1}{s+1}$ , so

$$S(j\omega) = \frac{1+j\omega}{11+j\omega}, \quad C(j\omega) = \frac{10}{11+j\omega}.$$

| $\omega$ | $ S(j\omega) $ | $ C(j\omega) $ |
|----------|----------------|----------------|
| 0.1      | 0.091          | 0.909          |
| 10       | 0.676          | 0.673          |
| 100      | 0.994          | 0.099          |

## 4.2 Intuition Add-on: What $S$ and $C$ feel like

### Everyday picture

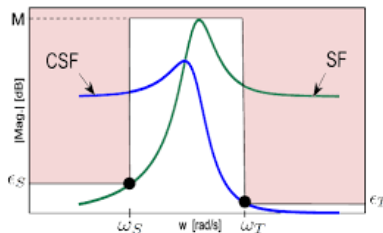
Where the loop has a lot of “leverage” (large  $|L|$ ), it fights errors strongly: *tracking* and *disturbance rejection* improve (small  $|S|$ ). But the same leverage also relays measurement wiggles (large  $|C|$ ). Where the loop has little leverage (small  $|L|$ ), noise does not get injected (small  $|C|$ ) but disturbances are less suppressed (large  $|S|$ ).

### DC checks (unity feedback,

$$L(0) = K$$

$$S(0) = \frac{1}{1+K}, \quad C(0) = \frac{K}{1+K}.$$

| $K$ | $S(0)$ | $C(0)$ |
|-----|--------|--------|
| 0.5 | 0.667  | 0.333  |
| 5   | 0.167  | 0.833  |
| 50  | 0.0196 | 0.980  |



## 4.2 Worked example: first-order plant (step-by-step)

Take  $G_c(s) = K$ ,  $G(s) = \frac{1}{s+1}$ ,  $H = 1$ .

$$L(s) = \frac{K}{s+1}, \quad S(s) = \frac{s+1}{s+1+K}, \quad C(s) = \frac{K}{s+1+K}.$$

**(A) Step tracking ( $R = 1/s$ ,  $T_d = N = 0$ ):**

$$E(s) = \frac{S(s)}{s} = \frac{s+1}{s(s+1+K)} = \frac{1}{K+1} \left( \frac{1}{s} + \frac{K}{s+K+1} \right).$$

$$\Rightarrow e(t) = \frac{1}{K+1} (1 + Ke^{-(K+1)t}), \quad e_{ss} = \frac{1}{K+1}.$$

*Sanity checks:*  $e(0^+) = 1$  (initially all error),  $e(\infty) = 1/(1+K)$ .

**(B) Step disturbance at plant input ( $T_d = D_0/s$ ,  $R = N = 0$ ):**

$$Y_d(s) = \frac{G}{1+L} T_d = \frac{1}{s+1+K} \cdot \frac{D_0}{s} \Rightarrow y_d(t) = \frac{D_0}{K+1} (1 - e^{-(K+1)t}).$$

|                                          | $K$ | $e_{ss}$ | $y_{d,ss}$ |
|------------------------------------------|-----|----------|------------|
| <b>Numbers (<math>D_0 = 0.2</math>):</b> | 0.5 | 0.667    | 0.133      |
|                                          | 5   | 0.167    | 0.033      |

## 4.2 Noise path: quick frequency check (use $K = 10$ )

With  $G_c = 10$ ,  $G = 1/(s + 1)$ :

$$S(j\omega) = \frac{1 + j\omega}{11 + j\omega}, \quad C(j\omega) = \frac{10}{11 + j\omega}.$$

A sinusoidal measurement noise  $N(t) = \sin(\omega t)$  is scaled at the error by  $|C(j\omega)|$  and at the reference path by  $|S(j\omega)|$ :

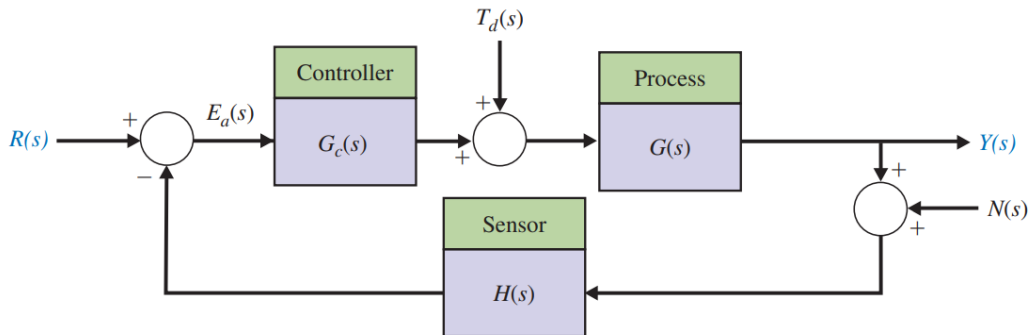
$$|S(j\omega)| = \frac{\sqrt{1 + \omega^2}}{\sqrt{11^2 + \omega^2}}, \quad |C(j\omega)| = \frac{10}{\sqrt{11^2 + \omega^2}}.$$

| $\omega$ | $ S(j\omega) $ | $ C(j\omega) $ |
|----------|----------------|----------------|
| 0.1      | 0.091          | 0.909          |
| 1        | 0.128          | 0.905          |
| 10       | 0.676          | 0.673          |
| 100      | 0.994          | 0.099          |

*Reading:* keep  $|L|$  high where disturbances live (typically low  $\omega$ ) and roll off at high  $\omega$  to avoid injecting sensor noise.

# Nonunity Feedback $H(s)$ (Sketch)

When  $H \neq 1$ , the algebra gives  $L = G_c GH$  and the same forms hold with  $H$  present.



## 4.3 Sensitivity: Normalized Definition (with intuition)

**Goal.** Quantify how much the closed-loop transfer  $T = Y/R$  changes when the plant  $G$  (or its parameter  $\alpha$ ) changes.

**Normalized (logarithmic) sensitivity:**

$$S_T^G \triangleq \frac{\partial \ln T}{\partial \ln G} = \frac{G}{T} \frac{\partial T}{\partial G} = \frac{1}{1 + L(s)} = S(s), \quad L = G_c G.$$

*Why logs?* A percentage change in  $G$  gives a percentage change in  $T$  scaled by  $S$ . If  $|S| \ll 1$ , closed loop is *insensitive*.

**Parameter (chain rule):**

$$S_T^\alpha \triangleq \frac{\partial \ln T}{\partial \ln \alpha} = S_T^G \cdot S_G^\alpha = S(s) \cdot \frac{\partial \ln G}{\partial \ln \alpha}.$$

**Example of  $S_G^\alpha$  (first-order plant)**

$G(s) = \frac{K_p}{\tau s + 1}$ :  $S_G^{K_p} = 1$  and  $S_G^\tau = \frac{\partial \ln G}{\partial \ln \tau} = -\frac{\tau s}{1 + \tau s}$  (high-frequency changes in  $\tau$  impact  $G$  more than low-frequency ones).

## Closed loop reduces sensitivity (derivation & low- $\omega$ view)

**Local variation model.** Let  $G \rightarrow G + \Delta G$  with only the reference active ( $T_d = N = 0$ ). Linearizing:

$$\Delta T \approx \frac{\partial T}{\partial G} \Delta G \Rightarrow \frac{\Delta T}{T} \approx S_T^G \frac{\Delta G}{G} = S \frac{\Delta G}{G}.$$

In the error channel (for a step), a convenient approximation is

$$\Delta E(s) \approx -\frac{\Delta G}{G} \frac{1}{L(s)} R(s), \quad |L| \gg 1, \quad H = 1,$$

which highlights the  $1/L$  suppression where the loop is strong.

**DC picture.** At  $s = 0$ ,  $S(0) = 1/(1 + L(0))$ : a 1 percent DC gain drift in  $G$  causes roughly  $S(0)$  percent drift in  $T$ .

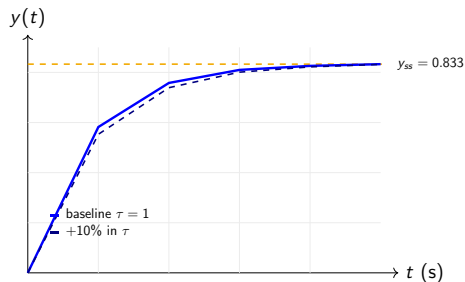
# Numeric example: drift in $K_p$ and $\tau$

Plant  $G = \frac{K_p}{\tau s + 1}$ , controller  $G_c = K$  (P), unity feedback. Closed-loop step response:

$$y(t) = \underbrace{\frac{KK_p}{1 + KK_p}}_{y_{ss}} \left(1 - e^{-t/\tau_{cl}}\right), \quad \tau_{cl} = \frac{\tau}{1 + KK_p}.$$

**Baseline:**  $K_p = 1$ ,  $\tau = 1$ ,  $K = 5 \Rightarrow y_{ss} = 0.833$ ,  $\tau_{cl} = 1/6$ .

**Perturb  $\tau$  by +10%:**  $\tau_{cl} = 1.1/6$  (slower), but *same*  $y_{ss}$ .

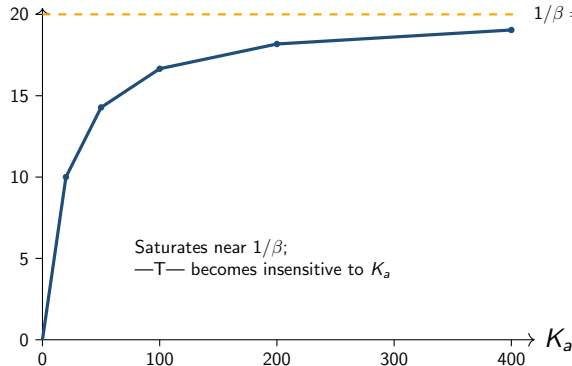


*Reading:* closed loop makes speed depend on  $\tau/(1 + KK_p)$  (reduced by  $1 + KK_p$ ) and keeps  $y_{ss}$

# Feedback amplifier — plots ( $\beta = 0.05$ )

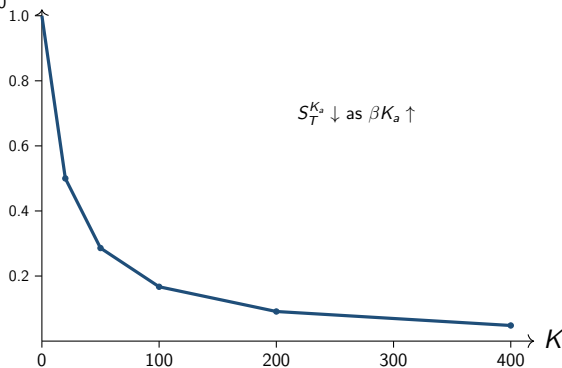
Closed-loop magnitude  $|T|$  vs  $K_a$

$|T|$  (V/V)



Sensitivity  $S_T^{K_a}$  vs  $K_a$

$S_T^{K_a}$



A 10% change in  $K_a$  produces a  $S_T^{K_a} \times 10\%$  change in  $T$ . Example: at  $K_a = 400$ ,  $S_T^{K_a} \approx 0.048 \Rightarrow$  about 0.48% change.

## 4.4 Disturbance rejection — what, where, why

**Setup (unity feedback).** Disturbance at plant input  $T_d$ :

$$E(s) = -S(s) G(s) T_d(s), \quad Y(s) = \frac{G}{1+L} T_d = S(s) G(s) T_d(s).$$

**Interpretation (plain English):**

- $S = \frac{1}{1+L}$  is the “leak.” Where  $|L|$  is large (low  $\omega$ ),  $|S|$  is small  $\Rightarrow$  disturbances are strongly rejected.
- The plant  $G$  shapes *which* frequencies of  $T_d$  reach  $Y$ ; feedback divides them by  $(1+L)$ .

**Other locations (for awareness):**

$$\text{Disturbance at plant } output: \quad Y = \frac{1}{1+L} T_{do} = S T_{do}, \quad E = \frac{1}{1+L} T_{do} = S T_{do}.$$

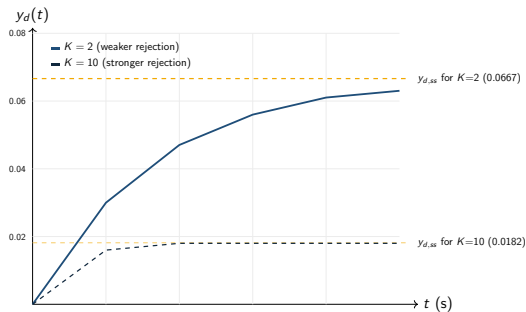
*Design idea:* Make  $|L|$  large over the disturbance spectrum (often low  $\omega$ ) while keeping margins acceptable.

# Disturbance rejection

Plant  $G = \frac{1}{s+1}$ , controller  $G_c = K$  (P), unity feedback. Input disturbance  $T_d = D_0/s$  (step),  $D_0 = 0.2$ :

$$Y_d(s) = \frac{G}{1+L} T_d = \frac{1}{s+1+K} \cdot \frac{D_0}{s} \Rightarrow y_d(t) = \frac{D_0}{1+K} \left(1 - e^{-(1+K)t}\right).$$

**Steady state:**  $y_{d,ss} = \frac{D_0}{1+K}$       **Time constant:**  $\tau_d = \frac{1}{1+K}$



*Reading:* larger  $K \Rightarrow$  smaller steady disturbance bias  $D_0/(1+K)$  and faster decay  $\tau_d=1/(1+K)$ .

# Measurement noise attenuation

Now let only sensor noise be active ( $R = T_d = 0$ ). With unity feedback:

$$E(s) = C(s) N(s), \quad Y(s) = -C(s) N(s), \quad C = \frac{L}{1 + L}.$$

## Interpretation:

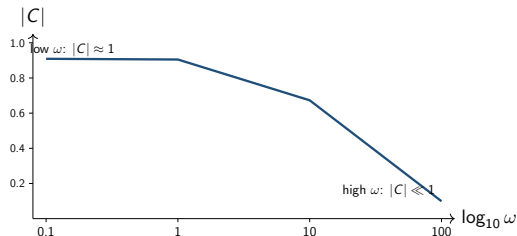
- $C$  tells how much measurement noise leaks into error/output. Where  $|L|$  is small (high  $\omega$ ),  $|C| \approx |L| \ll 1 \Rightarrow$  noise is *not* injected.
- Where  $|L|$  is large (low  $\omega$ ),  $|C| \approx 1 \Rightarrow$  low-frequency sensor bias will show up unless filtered.

*Design idea:* Keep high loop gain at low  $\omega$  for disturbances, but roll off  $|L|$  at high  $\omega$  to avoid noise injection.

## Noise path — numeric frequency view

Example ( $G_c = 10$ ,  $G = 1/(s + 1)$ ):  $C(j\omega) = \frac{10}{11 + j\omega}$ ,  $|C(j\omega)| = \frac{10}{\sqrt{11^2 + \omega^2}}$ .

| $\omega$ | $ C(j\omega) $ | comment                                                  |
|----------|----------------|----------------------------------------------------------|
| 0.1      | 0.909          | low- $\omega$ : noise passes (large $C$ )<br>still large |
| 1        | 0.905          |                                                          |
| 10       | 0.673          |                                                          |
| 100      | 0.099          | high- $\omega$ : noise attenuated                        |



*Rule-of-thumb:* high gain at low frequency; low gain at high frequency.

## Checkpoint 4.3 (Poll)

- 1 Write  $S_T^G$  for  $T = \frac{G}{1+G}$ .
- 2 If loop gain  $|G| \gg 1$ , is  $T$  more or less sensitive to  $G$ ?
- 3 Give one reason designers prefer high loop gain at low frequency.

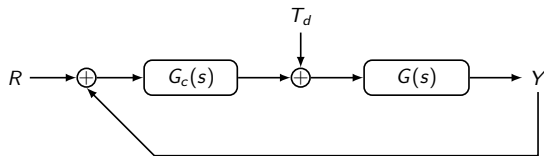
## Checkpoint 4.3 — Solutions

- 1)  $S_T^G = \frac{1}{1+G}$ .
- 2) Less sensitive (approaches 0).
- 3) Reduces sensitivity to plant variations and improves tracking/rejection.

# Where disturbances enter — with diagrams

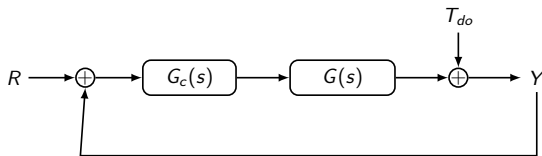
**Notation (unity unless shown):**  $L = G_c GH$ ,  $S = \frac{1}{1+L}$ ,  $C = \frac{L}{1+L}$ .

**A) Plant-input disturbance  $T_d$  (e.g., load torque)**



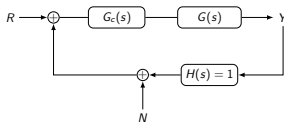
$$\text{Transfer: } Y = \frac{G}{1+L} T_d = S G T_d \quad \text{and} \\ E = -S G T_d.$$

**B) Plant-output disturbance  $T_{do}$  (e.g., vibration)**



$$\text{Transfer: } Y = \frac{1}{1+L} T_{do} = S T_{do} \quad \text{and} \\ E = S T_{do}.$$

**Sensor noise  $N$  adds in the feedback path (unity  $H$  shown explicitly):**



## Derivation (A): plant-input disturbance $T_d$

**Loop equations (unity  $H$ , show  $N$  for completeness):**

$$\begin{aligned} e &= r - y - n, & u &= G_c e, \\ y &= G(u + T_d). \end{aligned}$$

Set  $r = n = 0$  (disturbance-only):

$$y = G(G_c(-y) + T_d) \Rightarrow y(1 + G_c G) = G T_d.$$

**Therefore**

$$\boxed{\frac{Y}{T_d} = \frac{G}{1 + G_c G} = \frac{G}{1 + L} = S G, \quad \frac{E}{T_d} = -\frac{G}{1 + L} = -S G.}$$

*Reading:* Disturbance is shaped by the *process sensitivity*  $S G$ . Large  $|L|$  (low  $\omega$ )  $\Rightarrow$  small  $|S|$  and strong rejection.

## Derivation (B): plant-output disturbance $T_{do}$

**Loop equations (unity  $H$ ):**

$$y = G G_c (r - y - n) + T_{do}.$$

Set  $r = n = 0$ :

$$y(1 + G_c G) = T_{do}.$$

**Therefore**

$$\boxed{\frac{Y}{T_{do}} = \frac{1}{1 + G_c G} = \frac{1}{1 + L} = S, \quad \frac{E}{T_{do}} = \frac{1}{1 + L} = S.}$$

*Reading:* Output disturbance is filtered directly by  $S$  (no factor  $G$ ). Low-frequency rejection improves with larger  $|L|$ .

## Nonunity sensor $H(s)$ (for completeness)

With  $e = r - Hy - n$  and  $L = G_c GH$ , the results generalize to:

$$\frac{Y}{T_d} = \frac{G}{1+L} = S G, \quad \frac{Y}{T_{do}} = \frac{1}{1+L} = S, \quad \frac{Y}{N} = -\frac{L}{1+L} = -C.$$

*Design corollary:* The **same**  $L$  that rejects disturbances (via  $S$ ) will inject sensor noise (via  $C$ ) if made too large at high  $\omega$ .

# Frequency-domain intuition (disturbance paths)

**Plant-input disturbance:**  $\frac{Y}{T_d} = S G = \frac{G}{1 + L}$ .

- If  $|L(j\omega)| \gg 1$  (often low  $\omega$ ):  $\left| \frac{Y}{T_d} \right| \approx \left| \frac{G}{L} \right| = \left| \frac{1}{G_c H} \right|$  (strong rejection).
- If  $|L(j\omega)| \ll 1$  (high  $\omega$ ):  $\left| \frac{Y}{T_d} \right| \approx |G|$  (little rejection; plant dynamics dominate).

**Plant-output disturbance:**  $\frac{Y}{T_{do}} = S = \frac{1}{1 + L}$ .

- Low  $\omega$ :  $\left| \frac{Y}{T_{do}} \right| \approx \left| \frac{1}{L} \right|$  (good).
- High  $\omega$ :  $\left| \frac{Y}{T_{do}} \right| \approx 1$  (no help from feedback).

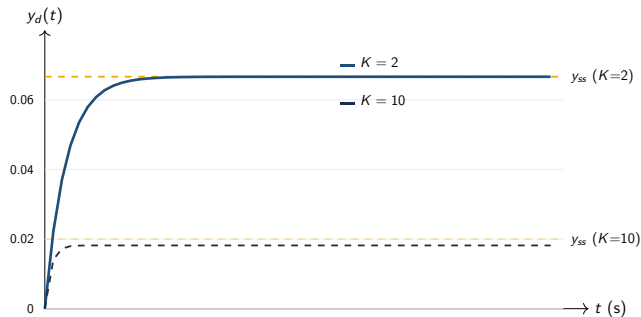
**Takeaway:** Shape  $L$  so that  $|L|$  is large over the *disturbance spectrum* (often low  $\omega$ ) and rolls off at high  $\omega$  to protect against noise.

# Numeric example A (plant-input): step disturbance, compact TikZ

Let  $G(s) = \frac{1}{s+1}$ ,  $G_c(s) = K$  (P), unity  $H$ . For  $T_d = D_0/s$ ,

$$\frac{Y}{T_d} = \frac{1}{s+1+K} \Rightarrow y_d(t) = \frac{D_0}{1+K} \left(1 - e^{-(1+K)t}\right).$$

Pick  $D_0 = 0.2$ , compare  $K = 2$  vs  $K = 10$ .



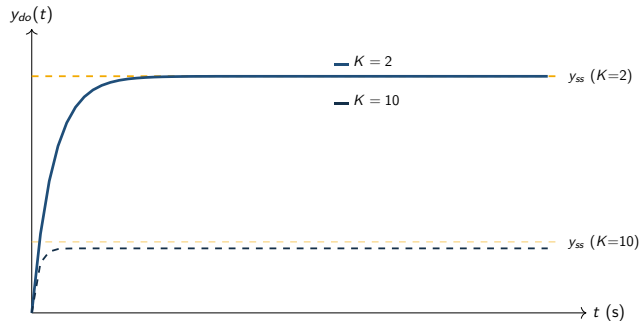
*Reading:* Higher  $K \Rightarrow$  smaller  $y_{ss} = D_0/(1+K)$  and faster  $\tau_d = 1/(1+K)$ .

## Numeric example B (plant-output): step disturbance, compact TikZ

With the same loop but disturbance *at the output*:  $T_{do} = D_0/s$  and

$$\frac{Y}{T_{do}} = \frac{1}{s + 1 + K} \Rightarrow y_{do}(t) = \frac{D_0}{1 + K} \left( 1 - e^{-(1+K)t} \right) \quad (\text{same form as input case}).$$

**But:** here the low- $\omega$  floor is  $\approx 1/|L|$  directly (no  $G$  factor).



*Reading:* For output disturbances,  $Y/T_{do} = S$ ; rejection tracks  $1/(1 + |L|)$  across frequency.

# Design checklist for disturbance rejection

- ① **Locate** the disturbance (input vs output vs sensor) and write  $Y/T$ .
- ② **Sketch** the spectrum of the disturbance; decide the needed  $|L|$  over that band.
- ③ **Shape**  $L$ : high at low  $\omega$  (disturbances), roll off at high  $\omega$  (noise). Keep phase margin  $\gtrsim 45^\circ$ .
- ④ **Include integral** if you need zero steady-state error to *step* disturbances.
- ⑤ **Check actuators**: saturation can undo rejection and slow decay.

## Poll Q1 — which transfer is correct (plant-input $T_d$ )?

For plant-input disturbance  $T_d$  (unity  $H$ ), the output due to  $T_d$  is:

①  $Y = \frac{L}{1+L} T_d$

②  $Y = \frac{1}{1+L} T_d$

③  $Y = \frac{G}{1+L} T_d$

④  $Y = L T_d$

**C)**  $Y = \frac{G}{1+L} T_d = S G T_d$  (derived by collecting  $y$  terms and solving).

## Poll Q2 — output disturbance path

For a disturbance *added to the plant output*, the output due to  $T_{do}$  is:

①  $Y = \frac{G}{1+L} T_{do}$

②  $Y = \frac{1}{1+L} T_{do}$

③  $Y = C T_{do}$

④  $Y = L T_{do}$

## Answer Q2

$$\mathbf{B)} \ Y = \frac{1}{1+L} T_{do} = S T_{do}.$$

## Poll Q3 — frequency strategy

If the disturbance power is concentrated below 2 rad/s and sensor noise dominates above 50 rad/s, then:

- ① Raise  $|L|$  across all frequencies for best results.
- ② Raise  $|L|$  below  $\approx 5$  rad/s and roll off above  $\approx 20$ – $30$  rad/s.
- ③ Keep  $|L|$  small everywhere to avoid noise injection.
- ④ None of the above.

**B)** Make  $|L|$  large over the disturbance band (low  $\omega$ ) and small at high  $\omega$  to attenuate sensor noise.

## Checkpoint 4.4 (Poll)

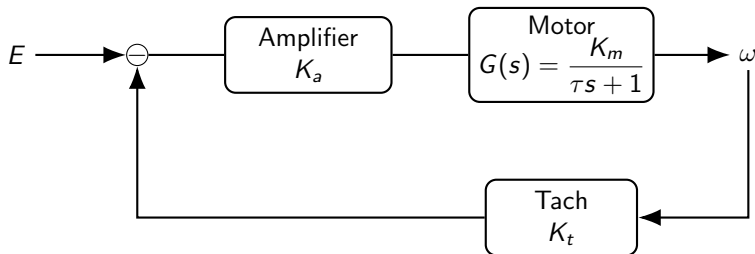
- ① How does high loop gain affect low-frequency disturbances?
- ② If a disturbance enters at the plant input, what factor multiplies it at the output?
- ③ Name one common disturbance in mechatronic systems.

## Checkpoint 4.4 — Solutions

- 1) It attenuates them (reduces their effect on output).
- 2)  $\frac{G}{1+G}$  (unity feedback case).
- 3) Load torque step, wind gust, supply ripple, sensor bias/noise, etc.

# Tachometer speed control — model and configuration

**Plant (speed dynamics):**  $G(s) = \frac{K_m}{\tau s + 1}$ ;    **Tachometer:**  $H(s) = K_t$  (voltage  $\propto$  speed).  
**Controller (amp):**  $G_c(s) = K_a$  (constant).    Negative feedback with tach signal.



*Textbook figure:* Insert the chapter's speed-control diagram here if desired.

# Derivation: closed-loop from command $E$ to speed $\omega$

**Loop algebra (unity sensor scaling shown explicitly):**

$$\omega = G K_a (E - K_t \omega) \Rightarrow \omega (1 + K_a K_t G) = K_a G E.$$

**Closed-loop transfer:**

$$\frac{\omega(s)}{E(s)} = \frac{K_a G(s)}{1 + K_a K_t G(s)} = \frac{K_a \frac{K_m}{\tau s + 1}}{1 + K_a K_t \frac{K_m}{\tau s + 1}} = \frac{K_a K_m}{\tau s + 1 + K_a K_t K_m}.$$

**Identify:**

$$\underbrace{\tau_{cl}}_{\text{time const.}} = \frac{\tau}{1 + K_a K_t K_m}, \quad \underbrace{K_{dc}}_{\text{steady gain}} = \frac{K_a K_m}{1 + K_a K_t K_m}.$$

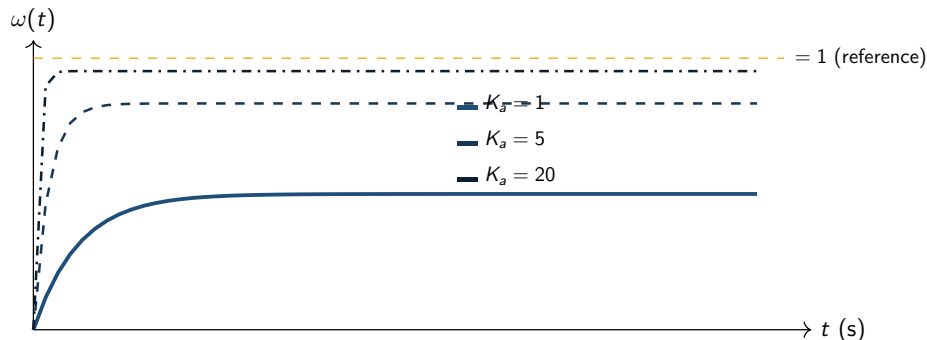
**Implications:** Increasing  $K_a$  (i) reduces  $\tau_{cl}$  (faster), and (ii) pushes  $K_{dc} \rightarrow 1/K_t$  as  $K_a \rightarrow \infty$ .

# Numerical example: faster transient with larger $K_a$

Choose  $K_m = 1$ ,  $K_t = 1$ ,  $\tau = 1$ . Then

$$\frac{\omega(s)}{E(s)} = \frac{K_a}{s + 1 + K_a}, \quad \omega(t) = \frac{K_a}{1 + K_a} \left(1 - e^{-(1+K_a)t}\right) E \text{ (unit } E\text{)}.$$

**Summary:**  $K_a = 1 \Rightarrow (\tau_{cl} = 0.5, \omega_\infty = 0.50)$ ;  $K_a = 5 \Rightarrow (0.167, 0.833)$ ;  
 $K_a = 20 \Rightarrow (0.048, 0.952)$ .



Reading: larger  $K_a \Rightarrow$  faster time constant and smaller steady tracking error to a unit command

# Recovering unity steady-state tracking (static calibration)

Because  $K_{dc} = \frac{K_a K_m}{1 + K_a K_t K_m} < 1$  (for  $K_t > 0$ ), we can pre-scale the reference by a constant  $K_r$ :

$$E' = K_r E, \quad K_r = \frac{1 + K_a K_t K_m}{K_a K_m}.$$

Then

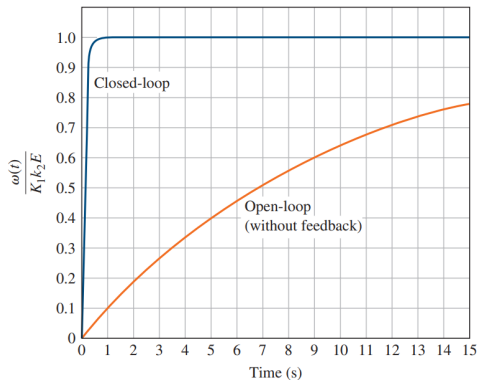
$$\frac{\omega(s)}{E(s)} = \underbrace{\frac{K_a K_m}{1 + K_a K_t K_m}}_{K_{dc}} \cdot K_r \cdot \frac{1 - e^{-(1+K_a K_t K_m)t/\tau}}{(\text{time part})} \Rightarrow \omega_\infty = 1 \cdot E.$$

**Key point:** calibration restores accurate steady-state tracking *without* sacrificing the faster pole (same  $\tau_{cl}$ ).

# Closed vs open loop step

**FIGURE 4.13**

The response of the open-loop and closed-loop speed control system when  $\tau = 10$  and  $K_1 K_a K_t = 100$ . The time to reach 98% of the final value for the open-loop and closed-loop system is 40 seconds and 0.4 seconds, respectively.



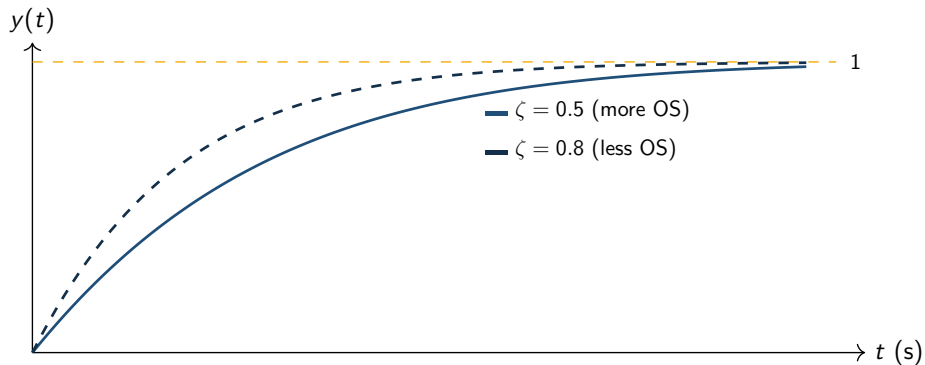
# From specs to $(\zeta, \omega_n)$ (for lightly damped 2nd-order)

Suppose the closed-loop is well-approximated by 2nd order with  $(\zeta, \omega_n)$ .

- 1 From %OS target, compute  $\zeta$  via  $\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$ .
- 2 From settling time  $T_s$  ( $\pm 2\%$ ), set  $\omega_n \approx \frac{4}{\zeta T_s}$ .
- 3 Shape  $L(s)$  (e.g., choose  $K_a$  and add a compensator zero/pole) so the dominant closed-loop pair lands at  $s_{1,2} \approx -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$ .

**Numeric example:** Target %OS = 10%  $\Rightarrow \zeta \approx 0.59$ ;  $T_s = 0.4$  s  $\Rightarrow \omega_n \approx \frac{4}{0.59 \times 0.4} \approx 16.95$  rad/s.

## Illustration: step responses for two $(\zeta, \omega_n)$



*Reading:* Higher  $\zeta$  (with similar  $\omega_n$ ) reduces overshoot and lengthens rise time;  $T_s$  scales with  $1/(\zeta\omega_n)$ .

- Increasing  $K_a$  speeds response but may reduce phase margin if unmodeled lags exist.
- Tachometer feedback passes low- $\omega$  sensor bias into steady-state unless you calibrate with  $K_r$ .
- Derivative (velocity) feedback reduces overshoot but amplifies high-frequency noise; use filters.
- Always verify actuator limits (saturation) — can cause longer  $T_s$  and larger error than predicted.

## Poll Q1 — time constant with tach feedback

For  $G(s) = \frac{K_m}{\tau s + 1}$ ,  $G_c = K_a$ ,  $H = K_t$ , the closed-loop time constant is:

- 1  $\tau$
- 2  $\frac{\tau}{1 + K_a K_t K_m}$
- 3  $\frac{1 + K_a K_t K_m}{\tau}$
- 4  $\frac{\tau}{1 + K_a}$

# Answer Q1

**B)**  $\tau_{cl} = \frac{\tau}{1 + K_a K_t K_m}$ . Derived from  $\frac{\omega}{E} = \frac{K_a K_m}{\tau s + 1 + K_a K_t K_m}$ .

## Poll Q2 — steady-state gain without calibration

With the same setup and unit  $E$ , the steady-state gain  $K_{dc}$  equals:

- ①  $K_a K_m$
- ②  $\frac{K_a K_m}{1 + K_a K_t K_m}$
- ③  $\frac{1 + K_a K_t K_m}{K_a K_m}$
- ④  $K_t$

## Answer Q2

$$\mathbf{B)} \ K_{dc} = \frac{K_a K_m}{1 + K_a K_t K_m}.$$

## Poll Q3 — calibration constant $K_r$

To recover unity steady-state tracking, choose  $K_r$  (pre-scaling on  $E$ ) as:

$$\textcircled{1} \quad K_r = \frac{K_a K_m}{1 + K_a K_t K_m}$$

$$\textcircled{2} \quad K_r = \frac{1 + K_a K_t K_m}{K_a K_m}$$

$$\textcircled{3} \quad K_r = K_t$$

$$\textcircled{4} \quad K_r = \frac{1}{K_t}$$

**B)**  $K_r = \frac{1 + K_a K_t K_m}{K_a K_m}$  so that  $K_{dc} \cdot K_r = 1$  while keeping  $\tau_{cl}$  unchanged.

## 4.6 Steady-state error — goals & roadmap

By the end, you can:

- Define steady-state error  $e_{ss}$  and compute it via the **final-value theorem**.
- Derive  $e_{ss}$  for **open loop** vs **closed loop** to a unit step input.
- Explain why closed-loop  $e_{ss}$  is *robust* to parameter drift while open loop is not.
- Size DC loop gain  $L(0)$  to meet an  $e_{ss}$  spec and check consequences.

# Definition and tool: final-value theorem

**Steady-state error** is the error after the transient dies out.

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s). \quad (\text{final-value theorem})$$

We will obtain  $E(s)$  from the loop equations for each case (open/closed) and then apply the limit.

# Open loop (no feedback): step-tracking error

**Open-loop error (unity sensor shown separately):**

$$E(s) = R(s) - Y(s) = (1 - G_c(s)G(s))R(s). \quad (\text{Eq. 4.46})$$

For a unit step  $R(s) = 1/s$ ,

$$e_{ss}^{\text{OL}} = \lim_{s \rightarrow 0} sE(s) = 1 - G_c(0)G(0). \quad (\text{from Eq. 4.49})$$

*Implication:* We can *calibrate*  $G_c(0)G(0) = 1$  to get  $e_{ss} = 0$ , but any drift in  $G$  spoils this.

## Closed loop (unity feedback): step-tracking error

With  $H(s) = 1$  and  $T_d = N = 0$ , the tracking error is

$$E(s) = \frac{1}{1 + G_c(s)G(s)} R(s) \triangleq S(s) R(s). \quad (\text{Eq. 4.47})$$

For a unit step  $R(s) = 1/s$ ,

$$e_{ss}^{\text{CL}} = \lim_{s \rightarrow 0} sE(s) = \frac{1}{1 + L(0)}, \quad L(0) = G_c(0)G(0). \quad (\text{Eq. 4.50})$$

*Implication:* Larger DC loop gain  $L(0) \Rightarrow$  smaller  $e_{ss}$ , and feedback keeps it small despite plant-parameter changes.

# Why closed loop wins: robustness to parameter drift

**Setup (textbook example):**  $G(s) = \frac{K}{\tau s + 1}$ ,  $G_c(s) = K_a$  (unity feedback).

- **Open loop:**  $e_{ss}^{OL} = 1 - KK_a$ ; calibrated to 0 when  $KK_a = 1$ . A 10% plant-gain drop causes  $e_{ss} = 0.1$ .
- **Closed loop:**  $e_{ss}^{CL} = \frac{1}{1 + KK_a} \Rightarrow$  with  $KK_a = 100$ ,  $e_{ss} = 1/101 \approx 0.0099$ . A 10% drop changes it to  $1/91 \approx 0.0110$  (only  $\approx 11 \times 10^{-4}$  increase).

*Takeaway:* Feedback maintains small  $e_{ss}$  over parameter drift; open loop does not.

# Step-by-step: compute $e_{ss}$ from loop algebra

1) Write error transfer:  $E(s) = \frac{1}{1 + G_c(s)G(s)} R(s)$  (unity  $H$ ).

2) Apply final-value theorem:

$$e_{ss} = \lim_{s \rightarrow 0} s E(s) = \lim_{s \rightarrow 0} \frac{s}{1 + G_c(s)G(s)} R(s).$$

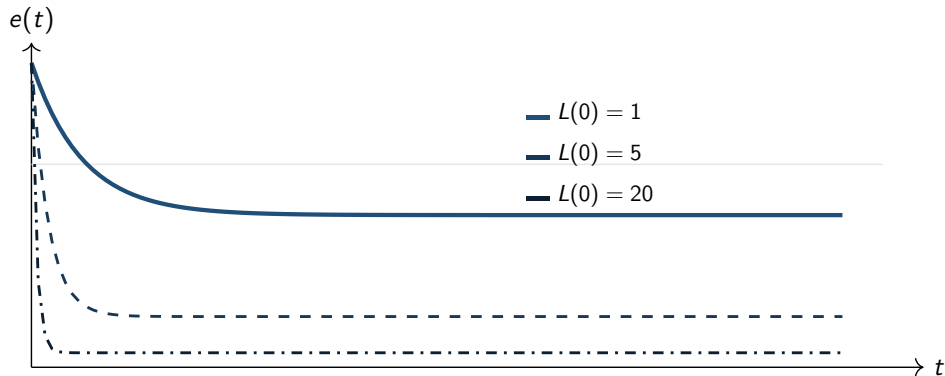
3) For a unit step  $R(s) = 1/s$ :

$$e_{ss} = \frac{1}{1 + G_c(0)G(0)} = \frac{1}{1 + L(0)}.$$

4) For general DC reference gain  $K_r$ :  $R(s) = \frac{K_r}{s} \Rightarrow e_{ss} = \frac{K_r}{1 + L(0)}$  (choose  $K_r$  to meet specs).

## Compact time plot: error decay for different $L(0)$

Unit step; with a first-order closed-loop pole at  $-(1 + L(0))/\tau$ , the error is  $e(t) = e_{ss} + (1 - e_{ss})e^{-(1+L(0))t/\tau}$ . (Illustrative.)



*Reading:* Larger  $L(0)$  gives *smaller* floor  $e_{ss}$  and *faster* decay.

# Summary

- Error signal, sensitivity, disturbance paths explain how feedback improves performance.
- Transient shaping: poles/zeros via controller; steady-state error via type and constants.

- PROBLEMS P4.7, P4.8, P4.11

Submit homework to **homeworkcut2026@126.com**; **due September 24, 2026.**