

Chapter 6 — Stability of Linear Feedback Systems

Dorf & Bishop, Modern Control Systems (14th ed.)

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After this chapter you should be able to:

- 1 Explain stability (BIBO, absolute vs. relative) and its pole/eigenvalue interpretations.
- 2 Construct the Routh array; apply Routh–Hurwitz (including special cases) to count RHP roots.
- 3 Assess relative stability via axis shifts and damping interpretation.
- 4 Determine stability for state-variable models via eigenvalues and determinants.
- 5 Apply methods to design examples (tracked vehicle, robot motorcycle) and with software.

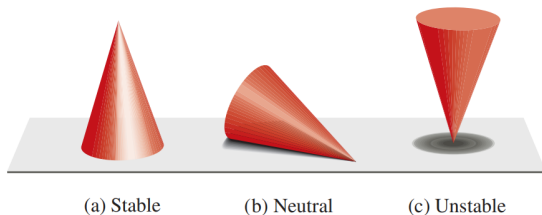
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Core Idea: What We Mean by Stability

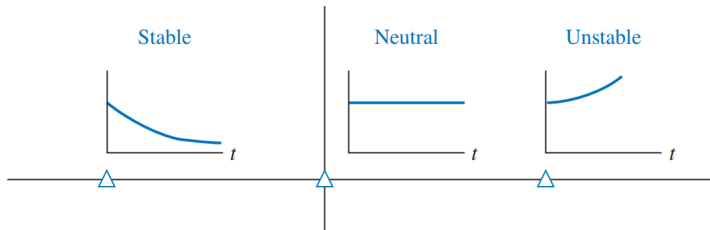
Definition. *BIBO stability*: bounded input $\Rightarrow$ bounded output. *Internal stability*: all natural modes decay.

Poles. For LTI systems, stability $\Leftrightarrow$ all poles of $T(s)$ in LHP (no RHP or repeated imaginary-axis poles).



Absolute vs. Relative Stability

- **Absolute:** Stable or not (all poles in LHP?).
- **Relative:** *How stable?* Measured by how far dominant poles are into the LHP (real part magnitude) or by damping ratio ζ of complex pairs.



Poles and Responses

For $T(s) = \frac{K \prod (s + z_k)}{\prod (s - p_i)}$, impulse response is sum of exponentials/oscillations from poles p_i :

$$y(t) = \sum_k A_k e^{\operatorname{Re}(p_k)t} \cos(\operatorname{Im}(p_k)t + \theta_k).$$

Stability condition: $\operatorname{Re}(p_i) < 0 \ \forall i$.

Marginal Cases

- Simple poles on $j\omega$ -axis $\Rightarrow$ sustained oscillations (marginally stable).
- Repeated poles on $j\omega$ -axis $\Rightarrow$ unbounded (unstable) response.

Why Routh–Hurwitz? (Motivation & Intuition)

Goal: Decide stability of $\Delta(s) = 0$ *without* solving for roots.

Idea: Build a tabular test (the Routh array) whose *first-column sign changes* equal the number of right-half-plane (RHP) roots.

Use cases: High-order polynomials, fast parametric sweeps, controller tuning inequalities, sanity checks before Root Locus/Nyquist.

Physical link: RHP roots $\Rightarrow$ modes that grow $\sim e^{+\sigma t}$; LHP roots $\Rightarrow$ decays; imaginary-axis roots $\Rightarrow$ sustained oscillations.

Setup: Characteristic Polynomial & Necessary Coefficient Tests

$$\Delta(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0, \quad a_n > 0.$$

- **Necessary (not sufficient):** all $a_i > 0$ if $a_n > 0$; any nonpositive $a_i \Rightarrow$ unstable.
- **Missing powers:** insert zero coefficients before building the array (e.g., no s^{n-2} term $\Rightarrow$ use $a_{n-2} = 0$).
- **Scaling invariance:** multiplying $\Delta(s)$ by a positive constant does not affect stability.

Building the Routh Array: Step-by-Step Algorithm

Input: $\Delta(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_0$ with $a_n > 0$.

Step 1 (Top two rows):

Row s^n : $a_n, a_{n-2}, a_{n-4}, \dots$

Row s^{n-1} : $a_{n-1}, a_{n-3}, a_{n-5}, \dots$

Step 2 (Next rows): For row s^{n-2} , entries b_k via 2×2 determinants divided by the leading element above:

$$b_k = \frac{\begin{vmatrix} a_{n-1} & a_{n-2k-1} \\ a_n & a_{n-2k} \end{vmatrix}}{a_{n-1}} = \frac{a_{n-1}a_{n-2k} - a_n a_{n-2k-1}}{a_{n-1}}.$$

Step 3: Similarly compute c_k (row s^{n-3}), d_k (row s^{n-4}), etc., until s^0 .

Step 4 (Decision): The number of sign changes in the *first column* equals the number of RHP roots. Zero sign changes $\Rightarrow$ all poles in LHP (stable), *subject to special-case handling below.*

Array Entries: Compact Formulas & Mnemonic

For $n \geq 3$:

$$b_1 = \frac{a_{n-1}a_{n-2} - a_na_{n-3}}{a_{n-1}}, \quad b_2 = \frac{a_{n-1}a_{n-4} - a_na_{n-5}}{a_{n-1}}, \dots$$

Next-row leading term:

$$c_1 = \frac{b_1a_{n-3} - a_{n-1}b_2}{b_1}.$$

Mnemonic: *(first of previous row) × (next of row above) – (first of row above) × (next of previous row), then divide by the previous-row leader.*

Worked Example 0 (Quick) — $n = 2$

$$\Delta(s) = a_2 s^2 + a_1 s + a_0, \quad a_2 > 0.$$

Stable iff $a_2 > 0$, $a_1 > 0$, $a_0 > 0$.

Why? The Routh array reduces to checking positivity of the three coefficients.

Worked Example 1 ($n = 3$) — Full Walkthrough

$\Delta(s) = s^3 + 4s^2 + 6s + 4$. Build the array:

Power	Coefficients		
s^3	1	6	0
s^2	4	4	0
s^1	$b_1 = \frac{4 \cdot 6 - 1 \cdot 4}{4} = 5$ $b_2 = 0$		
s^0	$c_1 = \frac{5 \cdot 4 - 4 \cdot 0}{5} = 4$		

First column: 1, 4, 5, 4 (no sign changes) $\Rightarrow$ **stable**.

Worked Example 2 ($n = 4$) — Symbolic Conditions

$$\Delta(s) = a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0, \quad a_4 > 0.$$

Top rows: $\{a_4, a_2, a_0\}$ and $\{a_3, a_1, 0\}$.

$$b_1 = \frac{a_3 a_2 - a_4 a_1}{a_3},$$

$$b_2 = a_0,$$

$$c_1 = \frac{b_1 a_1 - a_3 b_2}{b_1},$$

$$d_1 = a_0.$$

Stable iff $a_4 > 0$, $a_3 > 0$, $b_1 > 0$, $c_1 > 0$, $a_0 > 0$.

This yields the classical inequalities:

$$a_3 a_2 > a_4 a_1, \quad a_3 a_2 a_1 > a_4 a_1^2 + a_3^2 a_0,$$

in addition to $a_i > 0$.

Worked Example 3 ($n = 4$) — Numeric

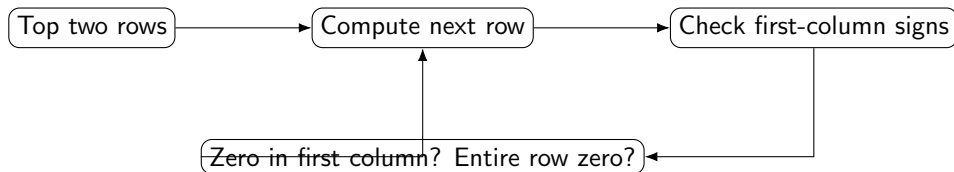
$$\Delta(s) = s^4 + 3s^3 + 3s^2 + 2s + 1.$$

Power	Coefficients		
s^4	1	3	1
s^3	3	2	0
s^2	$b_1 = \frac{3 \cdot 3 - 1 \cdot 2}{3} = \frac{7}{3}$ $b_2 = 1$		
s^1	$c_1 = \frac{(7/3) \cdot 2 - 3 \cdot 1}{7/3} = \frac{5}{7}$ $c_2 = 0$		
s^0	$d_1 = 1$		

First column: 1, 3, $7/3$, $5/7$, 1 (no sign changes) $\Rightarrow$ **stable**.

Special Cases: What Can Go Wrong?

- 1 **First-column zero** while row has other nonzeros $\Rightarrow$ use ε -replacement to proceed.
- 2 **Entire row zero** $\Rightarrow$ symmetric root pairs about the origin; build auxiliary $U(s)$ from the row above and replace with $U'(s)$.
- 3 **Imaginary-axis roots** $\Rightarrow$ marginal (simple) or unstable (repeated); Routh flags these via the zero-row situation.



Case 2: First-Column Zero (ε -Trick) with Example

$\Delta(s) = s^3 + 2s^2 + s + 2$. Top rows: $\{1, 1, 0\}$ and $\{2, 2, 0\}$.

Row s^1 leading term would be $b_1 = \frac{2 \cdot 1 - 1 \cdot 2}{2} = 0 \Rightarrow$ first-column zero.

Fix: Replace $0 \mapsto \varepsilon > 0$ for that leading entry, complete the array, count sign changes, then take $\varepsilon \rightarrow 0^+$.

Why it works: Avoids division by zero and preserves the limiting sign pattern.

Case 3: Entire Row of Zeros (Auxiliary Polynomial) — Example

Even polynomial with symmetric roots: $\Delta(s) = s^4 + 2s^2 + 1 = (s^2 + 1)^2$. Build Routh:

Power	Coefficients		
s^4	1	2	1
s^3	0	0	0 (entire row zero)

Row above corresponds to an *even-order* polynomial $U(s) = s^4 + 2s^2 + 1$.

Procedure: Replace the zero row with coefficients of $U'(s) = 4s^3 + 4s$, then continue Routh.

Result indicates *repeated* imaginary-axis roots at $\pm j \Rightarrow$ **unstable** (BIBO).

Normalized Form, Scaling, and Quick Tests

Normalization: divide by a_n and optionally scale by ω_n so $\Delta^*(s) = s^n + bs^{n-1} + cs^{n-2} + \dots + 1$.

Quick tests:

- $n = 2$: $a_2 > 0$, $a_1 > 0$, $a_0 > 0$.
- $n = 3$: $a_i > 0$ and $a_2a_1 > a_3a_0$.
- $n = 4$: $a_i > 0$, $a_3a_2 > a_4a_1$, and $a_3a_2a_1 > a_4a_1^2 + a_3^2a_0$.

Sanity: If any $a_i \leq 0$ (with $a_n > 0$) $\Rightarrow$ instability. If necessary tests pass but Routh shows a sign change, the system is unstable.

Common Pitfalls & How to Avoid Them

- ❶ **Forgetting zeros for missing powers:** always insert 0 coefficients.
- ❷ **Division by zero in array:** use the ε -trick for a zero *leading* element.
- ❸ **Ignoring entire zero rows:** must use auxiliary $U(s)$ and $U'(s)$.
- ❹ **Sign mistakes:** compute each 2×2 determinant carefully; keep a clean table.
- ❺ **Assuming positive coefficients are sufficient:** not true for $n \geq 3$.

Guided Practice (Try Now)

- 1 Test stability of $s^3 + 2s^2 + 4s + K$; derive the allowable K range.
- 2 For $\Delta(s) = s^4 + 2s^3 + 5s^2 + 6s + K$, find inequalities on K for stability.
- 3 Detect and resolve special cases for $\Delta(s) = s^4 + 2s^2 + 1$ (what does Routh reveal?).

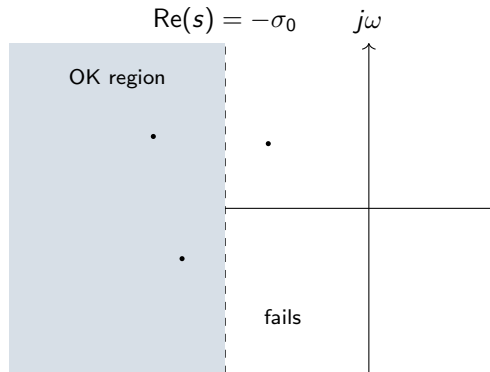
Tip: Do necessary coefficient checks, then Routh, then (if desired) the axis-shift test of Sec. 6.3 for *relative* stability.

What is Relative Stability? (s-Plane View)

Absolute stability: are all poles in LHP?

Relative stability: *how far* are poles into the LHP? Further left $\Rightarrow$ faster decay, smaller overshoot.

- Choose a vertical line $\text{Re}(s) = -\sigma_0$.
- If *all* poles lie left of this line, the response envelope decays at least as $e^{-\sigma_0 t}$.
- Larger $\sigma_0 \Rightarrow$ stricter requirement $\Rightarrow$ faster guaranteed decay.



Axis-Shift Test: Routh with $s = s_n - \sigma_0$

Goal: certify that all poles lie to the left of $\text{Re}(s) = -\sigma_0$ *without* solving for roots.

Procedure (algorithmic):

- 1 Form the **shifted** polynomial $\Delta_{\sigma_0}(s_n) = \Delta(s_n - \sigma_0)$.
- 2 Build the Routh array of $\Delta_{\sigma_0}(s_n)$ (insert zero coefficients if powers are missing).
- 3 **Decision:** If the first column has *no* sign changes, then all original poles satisfy $\text{Re}(s) < -\sigma_0$.

Interpretation: Move the imaginary axis to $\text{Re}(s) = -\sigma_0$ and apply absolute stability to the shifted system.

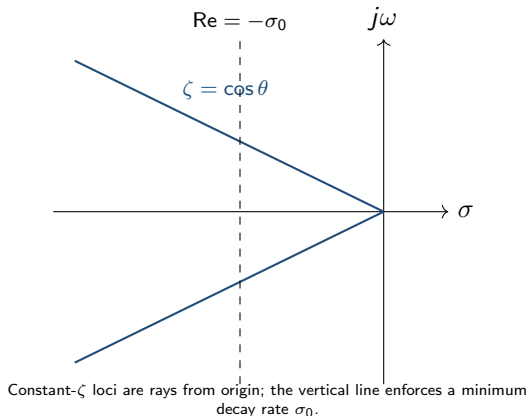
Choosing σ_0 from Time-Domain Specs

Decay/time-constant:

- Envelope $\sim e^{-\sigma t}$; pick $\sigma_0 \approx 1/\tau$ for desired time constant τ .
- 2% settling time (dominant real/adequately damped): $T_s \approx \frac{4}{\sigma_0}$.

Overshoot (2nd-order dominant):

- $M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$; e.g., $M_p = 10\% \Rightarrow \zeta \approx 0.59$.
- $\sigma = \zeta\omega_n$; ensure $\zeta\omega_n \geq \sigma_0$.



Worked Example 1: Certify a Minimum Decay Rate

Given $\Delta(s) = s^3 + 4s^2 + 6s + 4$ (stable), certify decay rate $\sigma_0 = 1$.

Step 1: Substitute $s = s_n - 1$,

$$\Delta_1(s_n) = (s_n - 1)^3 + 4(s_n - 1)^2 + 6(s_n - 1) + 4 = s_n^3 + s_n^2 + s_n + 1.$$

Step 2: Routh array of $s_n^3 + s_n^2 + s_n + 1$:

Power	Coefficients		
s_n^3	1	1	0
s_n^2	1	1	0
s_n^1	$b_1 = \frac{1 \cdot 1 - 1 \cdot 1}{1} = 0$		0
s_n^0	use ε trick to continue		

Step 3: Replace the zero leading element in row s_n^1 by $\varepsilon > 0$ and complete the table. No sign change in the first column as $\varepsilon \rightarrow 0^+$.

Conclusion: All poles satisfy $\text{Re}(s) < -1$ (guaranteed decay rate ≥ 1).

Worked Example 2 (Outline): Largest Admissible σ_0

Polynomial $\Delta(s) = s^4 + 3s^3 + 3s^2 + 2s + 1$. Find the largest σ_0 such that $\operatorname{Re}(s) < -\sigma_0$.

Recipe:

- 1 Compute $\Delta_{\sigma_0}(s_n) = \Delta(s_n - \sigma_0)$ (symbolic in σ_0).
- 2 Build its Routh array. Require every first-column entry > 0 .
- 3 Solve the resulting inequalities for $\sigma_0 > 0$ to get $0 < \sigma_0 < \sigma_0^{\max}$.

Comment: Picking σ_0 near $\sigma_0^{\max}$ yields the fastest certified decay consistent with this model.

Interpreting Damping (ζ) and Geometry

For a complex pair $s_{1,2} = -\zeta\omega_n \pm j\omega_d$ with $\omega_d = \omega_n\sqrt{1-\zeta^2}$:

- **Decay rate:** $\sigma = \zeta\omega_n$. Vertical line test enforces $\zeta\omega_n \geq \sigma_0$.
- **Overshoot:** $M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$ (2nd-order approx.). Targeting M_p sets ζ .
- **Settling time:** $T_s(2\%) \approx \frac{4}{\zeta\omega_n}$ for $\zeta \gtrsim 0.5$. Combine with $\zeta\omega_n \geq \sigma_0$ to choose σ_0 .

Design Recipe: Using Relative Stability in Tuning

- 1 Convert time-domain specs to (σ_0, ζ) : e.g., $T_s \leq T^* \Rightarrow \sigma_0 \geq 4/T^*$; $M_p \leq M_p^* \Rightarrow \zeta \geq \zeta^*$.
- 2 Substitute $s = s_n - \sigma_0$ and build Routh for $\Delta_{\sigma_0}(s_n)$.
- 3 Derive **inequalities** in controller parameters that make the first column strictly positive.
- 4 (Optional) Enforce ζ by shaping the dominant quadratic factor or using root-locus geometry.
- 5 Validate numerically (poles/step) to confirm decay rate and overshoot.

Common Pitfalls (Relative Stability)

- Choosing σ_0 too large $\Rightarrow$ infeasible inequalities (no solution).
- Forgetting to insert zero coefficients after the shift (powers change!).
- Misreading zero rows: apply auxiliary polynomial and continue before concluding.
- Confusing decay-rate guarantee with damping ratio: vertical line enforces σ only; use ζ geometry for overshoot.

Guided Practice (Relative Stability)

- 1 For $\Delta(s) = s^3 + 5s^2 + 7s + K$ with $K = 10$, find the largest σ_0 such that all poles lie left of $\text{Re} = -\sigma_0$.
- 2 Specs: $T_s \leq 1.5\text{ s}$ and $M_p \leq 10\%$. Choose (σ_0, ζ) and describe how you certify them via axis-shift Routh and pole plots.

State-Variable Stability: Big Picture

Model: $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t) + Du(t)$.

Zero-input (internal) dynamics: set $u \equiv 0 \Rightarrow x(t) = e^{At}x(0)$.

Main criterion (continuous-time LTI):

Asymptotically stable $\iff$ all eigenvalues of A satisfy $\operatorname{Re} \lambda_i < 0$.

Equivalently: the characteristic polynomial $\chi_A(\lambda) = \det(\lambda I - A)$ has all roots in the open LHP.

Connection: For a *minimal* realization, the poles of $G(s)$ coincide with the eigenvalues of A .

Why Eigenvalues? From A to e^{At}

$$x(t) = e^{At}x(0), \quad e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}.$$

If $A = VJV^{-1}$ (Jordan form) with blocks $J_i = \lambda_i I + N_i$ (N_i nilpotent), then

$$e^{At} = V e^{Jt} V^{-1}, \quad e^{J_i t} = e^{\lambda_i t} \left(I + N_i t + \frac{1}{2!} N_i^2 t^2 + \dots \right).$$

Implications:

- $\operatorname{Re} \lambda_i < 0 \Rightarrow e^{\lambda_i t} \rightarrow 0$ even with Jordan chains.
- $\operatorname{Re} \lambda_i > 0 \Rightarrow$ an exponentially growing term $\Rightarrow$ unstable.
- $\operatorname{Re} \lambda_i = 0$: simple $\pm j\omega$ is marginal; *defective* (Jordan size > 1) gives polynomial growth ($te^{j\omega t}$) $\Rightarrow$ unstable.

Internal Stability Types

- **Asymptotically stable:** all $\text{Re } \lambda_i < 0$; $x(t) \rightarrow 0$ for all $x(0)$.
- **Marginally stable:** all $\text{Re } \lambda_i \leq 0$, at least one $\text{Re} = 0$, and *all* imaginary-axis eigenvalues are simple (no Jordan chains).
- **Unstable:** any $\text{Re } \lambda_i > 0$; or any imaginary-axis eigenvalue with Jordan size > 1 .

BIBO vs. internal: For *minimal* realizations of proper $G(s)$,

Internal stability $\iff$ BIBO stability $\iff$ all poles (eigs of A) in LHP.

Nonminimal realizations may hide unstable uncontrollable/unobservable modes (internally unstable but BIBO-stable transfer).

Characteristic Equation & Transfer Function

$$\chi_A(\lambda) = \det(\lambda I - A) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0.$$

$$G(s) = C(sI - A)^{-1}B + D = \frac{N(s)}{\det(sI - A)}.$$

Key link: Poles of $G(s)$ are roots of $\det(sI - A) = 0$; under *minimality*, these are the eigenvalues of A .

Practice: Compute stability by (i) eigenvalues of A , or (ii) Routh–Hurwitz on $\chi_A(s)$.

Worked Derivation: 2×2 Case (Step-by-Step)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \chi_A(\lambda) = \det(\lambda I - A) = (\lambda - a)(\lambda - d) - bc:$$

$$\chi_A(\lambda) = \lambda^2 - (a + d)\lambda + (ad - bc).$$

By $n=2$ Routh/Hurwitz, **stable** $\iff$ all coefficients > 0 :

$$\lambda^2 + \underbrace{(-(a + d))}_{>0} \lambda + \underbrace{(ad - bc)}_{>0}.$$

Thus,

$$\boxed{\operatorname{tr}(A) = a + d < 0 \quad \text{and} \quad \det(A) = ad - bc > 0}.$$

Geometric view: sum of eigenvalues negative and product positive $\Rightarrow$ both eigenvalues in LHP.

Worked Derivation: 3×3 via Routh

Let $\chi_A(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$. Routh conditions for LHP:

$$a_2 > 0, \quad a_1 > 0, \quad a_0 > 0, \quad a_2 a_1 > a_0.$$

Routh table (details):

Power	Coefficients		
λ^3	1	a_1	0
λ^2	a_2	a_0	0
λ^1	$\frac{a_2 a_1 - a_0}{a_2}$	0	
λ^0	a_0		

Compute a_i by expanding $\det(\lambda I - A)$ (or by Faddeev–Leverrier).

Numeric Example: From A to Stability (3×3)

$$A = \begin{bmatrix} -2 & 1 & 0 \\ -3 & -1 & 0 \\ 0 & 1 & -4 \end{bmatrix}, \quad \lambda I - A = \begin{bmatrix} \lambda + 2 & -1 & 0 \\ 3 & \lambda + 1 & 0 \\ 0 & -1 & \lambda + 4 \end{bmatrix}.$$

Expand along the 3rd column:

$$\chi_A(\lambda) = (\lambda + 4)((\lambda + 2)(\lambda + 1) + 3) = (\lambda + 4)(\lambda^2 + 3\lambda + 5).$$

Roots: $\lambda = -4$ and $\lambda = \frac{-3 \pm j\sqrt{11}}{2} \Rightarrow$ all in LHP $\Rightarrow$ **asymptotically stable**.

Special Structures & Quick Checks

- **Diagonal A :** eigenvalues are the diagonal entries; LHP check is immediate.
- **Triangular A :** eigenvalues are diagonal entries.
- **Jordan block at $j\omega$:** if block size > 1 , e^{At} contains $te^{j\omega t}$ terms $\Rightarrow$ unstable (internal).
- **Nearly defective A :** eigenvalue computation is sensitive; Routh on χ_A can be numerically steadier.

Minimality, Hidden Modes, and BIBO

- If (A, B, C) is *not* minimal, $G(s)$ may be stable while A has unstable uncontrollable/unobservable modes (hidden internally).
- Kalman decomposition reveals controllable/observable subspaces; internal stability requires *all* eigenvalues of A in LHP, not just the pole set of $G(s)$.
- For transfer-based checks, ensure a minimal realization to equate internal and BIBO stability.

Canonical Forms: Fast Route to χ_A

Controllable canonical form for $\chi(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix},$$

whose eigenvalues are exactly the roots of $\chi(s)$ by construction.

Observable canonical form: similar convenience for reading $\chi(s)$ off A .

Relative Stability in State Space

- **Minimum decay rate:** require $\operatorname{Re} \lambda_i \leq -\sigma_0$. Equivalent test: apply Routh to $\chi_A(s + \sigma_0)$.
- **Dominant pair:** if $\lambda_{1,2} = -\zeta\omega_n \pm j\omega_d$, then decay rate is $\zeta\omega_n$; overshoot from $M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$.

Practical Workflow for State-Space Stability

- 1 Derive A from physics or block diagrams.
- 2 Compute $\chi_A(\lambda)$ (symbolically or numerically) or $\text{eig}(A)$.
- 3 Check $\text{Re } \lambda_i$ or use Routh on χ_A .
- 4 If margins are required: test $\chi_A(s + \sigma_0)$ for a chosen $\sigma_0 > 0$.
- 5 Validate by simulation: $x(t) = e^{At}x(0)$, step response, pole plots.

Template: Second-Order State Matrix

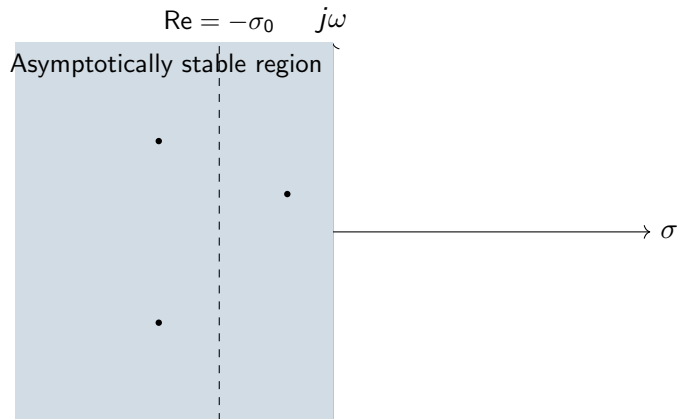
$$A = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix}$$

$$\chi_A(\lambda) = \lambda^2 + 2\zeta\omega_0 \lambda + \omega_0^2, \quad \lambda_{1,2} = -\zeta\omega_0 \pm j\omega_0\sqrt{1 - \zeta^2}.$$

Stable iff $\zeta > 0$ and $\omega_0 > 0$ (both eigenvalues in LHP).

Relative stability: decay rate $\sigma = \zeta\omega_0$; settling $\approx 4/(\zeta\omega_0)$ for $\zeta \gtrsim 0.5$.

Eigenvalues in the s -Plane (TikZ)



Practice: Quick Checks

- 1 $A = \begin{bmatrix} -1 & 2 \\ -3 & -4 \end{bmatrix}$. Use trace/determinant to decide stability.
- 2 For $\chi_A(\lambda) = \lambda^3 + 6\lambda^2 + 11\lambda + 6$, use the $n=3$ Routh test.
- 3 Suppose A has an eigenvalue at $j2$ with a size-2 Jordan block. Classify stability.

Poll Q1: Internal Stability Test

For $\dot{x} = Ax$, the system is asymptotically stable iff:

- ① $\det(A) > 0$.
- ② A is diagonalizable.
- ③ All eigenvalues of A have strictly negative real parts.
- ④ $\text{tr}(A) < 0$.

Poll Q2: 2×2 Criterion

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, a necessary and sufficient condition for LHP eigenvalues is:

- ① $a < 0$ and $d < 0$.
- ② $b > 0$ and $c > 0$.
- ③ $\text{tr}(A) < 0$ and $\det(A) > 0$.
- ④ $\det(A) < 0$.

Poll Q3: Minimality

Which statement is true?

- ① If $G(s)$ is stable, then A must be stable for any realization.
- ② If the realization is minimal and $G(s)$ is stable, then A is internally stable.
- ③ If A is unstable but uncontrollable, the system is internally stable.
- ④ Minimality has no relation to internal stability.

Poll Q4: Jordan Block on $j\omega$

If A has a Jordan block of size 2 at $\lambda = j\omega$ ($\omega \neq 0$), the system is:

- ① Asymptotically stable.
- ② Marginally stable.
- ③ Unstable (internal).
- ④ BIBO stable but internally unstable for any realization.

Poll Q5: Relative Stability by Shift

To guarantee $\text{Re } \lambda_i \leq -\sigma_0$ for all eigenvalues, you can:

- ① Check $\chi_A(s)$ with standard Routh only.
- ② Apply Routh to $\chi_A(s + \sigma_0)$ and require a positive first column.
- ③ Shift A to $A + \sigma_0 I$ and require eigenvalues > 0 .
- ④ None of the above.

Poll Answers (Sec. 6.4)

Q1: (3) Q2: (3) Q3: (2) Q4: (3) Q5: (2)

Summary

- Stability $\leftrightarrow$ LHP poles; use the characteristic equation to test.
- Routh–Hurwitz gives stability and RHP root count without solving roots.

- EXERCISES E6.13, E6.15, E7.13, E7.26
- PROBLEMS P6.5, P6.13,

Submit homework to **homeworkscut2026@126.com**; **due Oct. 15, 2026.**