

Chapter 7 — The Root-Locus Method (Sketching & Design)

Dorf & Bishop, Modern Control Systems (14th ed.)

Gang Chen

September 30, 2026

7.1 Introduction — Why Root Locus? (Big Picture)

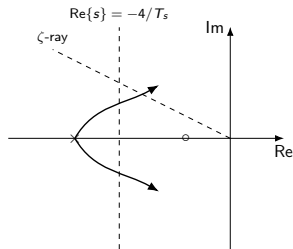
What we care about. Closed-loop performance is determined by the locations of the system's *characteristic roots* (closed-loop poles) in the s -plane:

- **Real part** ($\text{Re}\{s\}$) controls speed/settling: more negative $\Rightarrow$ faster decay.
- **Imag part** ($\text{Im}\{s\}$) controls oscillation frequency and overshoot.

Root locus (RL). The continuous *paths* traced by the closed-loop poles as a single scalar parameter (usually a gain K) varies from 0 to ∞ .

Why use RL?

- **Fast stability insight:** see if poles cross into $\text{Re}\{s\} > 0$ as K changes.
- **Transient design:** place poles near desired damping-ratio rays and settling-time lines.
- **Controller tuning:** pick K (and zero locations) with geometric intuition before fine tuning.



7.2 Root Locus Concept — Setup & Key Equations

Unity-feedback loop: $T(s) = \frac{KG(s)}{1 + KG(s)}$. The **characteristic equation** (CE) is

$$1 + KG(s) = 0 \iff KG(s) = -1.$$

Convert to polar form:

$$|KG(s)| = 1, \quad \angle(KG(s)) = 180^\circ + 360^\circ k, \quad k \in \mathbb{Z}.$$

Because $K > 0$, $\angle K = 0$, so the *angle condition* and *magnitude condition* become

$$\boxed{\angle G(s) = 180^\circ + 360^\circ k} \quad \text{and} \quad \boxed{K = \frac{1}{|G(s)|}}.$$

Definition (working). The root locus is the set of s in the complex plane that satisfy the *angle* rule as K varies; the *magnitude* then reads off the corresponding K .

Tip. Always check the angle rule first (geometry), then compute K (number).

Second-Order Illustration — Minimal Algebra & Geometry

Example: $G(s) = \frac{1}{s(s+2)}$, unity feedback.

$$1 + \frac{K}{s(s+2)} = 0 \iff s^2 + 2s + K = 0.$$

Closed-form roots: $s_{1,2} = -1 \pm \sqrt{1-K}$.

- $0 < K < 1 \Rightarrow$ two distinct *real* poles in $(-2, 0)$ approaching each other.
- $K = 1 \Rightarrow$ double pole at $s = -1$ (**breakaway** point).
- $K > 1 \Rightarrow$ complex-conjugate poles with *fixed* real part -1 (vertical motion).

Real-axis membership (fix a subtlety!). Poles at 0 and $-2 \Rightarrow$ the *only* real-axis RL segment is $(-2, 0)$:

Interval	# of real poles/zeros to the right
$(-\infty, -2)$	2 (even) $\Rightarrow$ off
$(-2, 0)$	1 (odd) $\Rightarrow$ on
$(0, \infty)$	0 (even) $\Rightarrow$ off

This matches the exact algebra: the real roots live in $(-2, 0)$ until $K = 1$, then go complex.

Angle & Magnitude Criteria for Factored $G(s)$

Write

$$G(s) = \frac{\prod_{i=1}^M (s - z_i)}{\prod_{j=1}^n (s - p_j)}.$$

At any test point s^* ,

$$\angle G(s^*) = \sum_{i=1}^M \angle(s^* - z_i) - \sum_{j=1}^n \angle(s^* - p_j) \stackrel{!}{=} 180^\circ + 360^\circ k.$$

If the angle test passes, recover the gain:

$$K = \frac{1}{|G(s^*)|} = \frac{\prod_{j=1}^n |s^* - p_j|}{\prod_{i=1}^M |s^* - z_i|}.$$

Practical guardrails.

- Use principal angles (e.g., $(-180^\circ, 180^\circ]$) and then reduce modulo 360° .
- Draw vectors from each pole/zero to s^* ; *subtract* pole angles from zero angles.
- For real-axis points the angles collapse to 0° or 180° , making hand checks quick.

Multiloop View & Equivalent $F(s)$ (When G is not “plain”)

For complicated loops, reduce to a *single* loop function $F(s)$:

$$\Delta(s) = 1 + F(s) = 0, \quad F(s) = K \frac{\prod(s - z_i)}{\prod(s - p_j)}.$$

Same rules: solve $F(s) = -1 \Rightarrow |F(s)| = 1, \angle F(s) = 180^\circ + 360^\circ k$.

Mini-example (sensor gain). If $F(s) = K G(s)H(s)$ with $H(s)$ fixed, you may view the open-loop as $G_{\text{eq}}(s) = G(s)H(s)$ and sweep K . Pole/zero *geometry* of G_{eq} controls the locus; K only scales magnitude.

Takeaway. Regardless of internal details, the root-locus geometry depends on *where* the open-loop poles/zeros are.

Checkpoint 7.1 (Poll) — Concept Essentials

1. Where do loci start when $K = 0$ and why?
2. What two tests characterize a locus point s^* ?
3. Why must a root locus drawn for real-coefficient systems be symmetric about the real axis?

Hints.

- Think “characteristic roots of $1 + KG(s) = 0$ as $K \rightarrow 0^+$ ”.
- Angle gives *geometry*; magnitude gives the *value of K* .
- Conjugation property of polynomials with real coefficients.

Checkpoint 7.1 — Detailed Solutions (with Notes)

1. **At the open-loop poles.** When $K \rightarrow 0^+$ the CE approaches $1 = 0$ unless $G(s) \rightarrow \infty$; this happens at poles of G where closed-loop poles must begin.
2. **Angle & Magnitude tests.** Angle: $\angle G(s^*) = 180^\circ + 360^\circ k$ ensures the point lies on the locus. Magnitude: $K = 1/|G(s^*)|$ then reads the gain.
3. **Symmetry.** Real-coefficient polynomials have complex roots in conjugate pairs; thus if s is a pole then $\bar{s}$ is too $\Rightarrow$ geometric symmetry about the real axis.

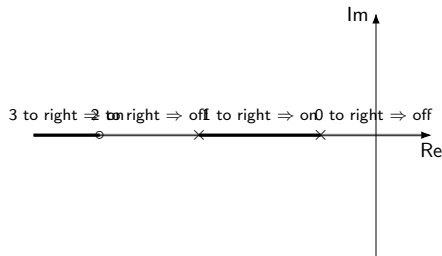
Rule Set (High-Level) — What to Draw First (Corrected Plot)

Starts/Ends. For $L_0(s) = K \frac{\prod(s-z_i)}{\prod(s-p_j)}$ with $K \geq 0$:

- n branches start at the n open-loop poles p_j ($K \rightarrow 0^+$).
- m branches end at the m open-loop zeros z_i ($K \rightarrow \infty$).
- If $n > m$, $n - m$ branches head to ∞ along straight-line *asymptotes*.

Symmetry. Mirror symmetry about the real axis.

Real-axis membership. A real point is on the locus iff the number of *real* poles+zeros to its right is **odd**.



Check: With marks at -5 ($\circ$), -3.2 ($\times$), -1 ($\times$), the “right-count” over the four real intervals is $\{3, 2, 1, 0\}$, giving ON–OFF–ON–OFF as plotted.

Why the Real-Axis Rule is “Odd” — One-Line Proof

For $s^* \in \mathbb{R}$,

$$\angle(s^* - a) = \begin{cases} 0^\circ, & s^* > a \\ 180^\circ, & s^* < a \end{cases}$$

so

$$\angle G(s^*) = 180^\circ \times (\# \text{ real poles/zeros to the right of } s^*) \pmod{360^\circ}.$$

The angle condition requires $\angle G(s^*) = 180^\circ \pmod{360^\circ}$, i.e., the count is **odd**. □

Common pitfall. Counting complex conjugate *pairs* that are *not* on the real line—they do *not* contribute 180° each to the real-axis test.

Asymptotes (for $n > m$) — Angles & Centroid Made Intuitive

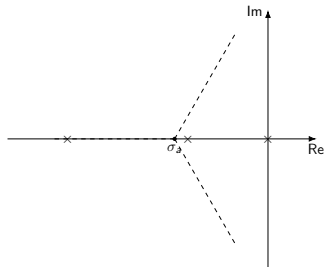
Angles (evenly spaced):

$$\theta_q = \frac{(2q+1)\pi}{n-m}, \quad q = 0, 1, \dots, n-m-1.$$

Centroid (where they emanate):

$$\sigma_a = \frac{\sum p_j - \sum z_i}{n-m} \quad (\text{conjugate pairs contribute twice their real part}).$$

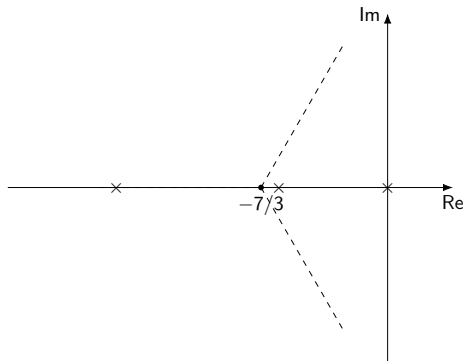
Intuition. Far from all poles/zeros, the net phase looks like that of $K(s - \sigma_a)^{m-n}$, which produces straight rays with those angles starting from the centroid.



Worked Example — Asymptotes Cleanly Computed

$$L_0(s) = \frac{1}{s(s+2)(s+5)}: p = \{0, -2, -5\}, \text{ no zeros} \Rightarrow n = 3, m = 0.$$

$$\sigma_a = \frac{0 + (-2) + (-5) - 0}{3} = -\frac{7}{3} \approx -2.333, \quad \theta_q = \frac{(2q+1)\pi}{3} \Rightarrow \{60^\circ, 180^\circ, 300^\circ\}.$$

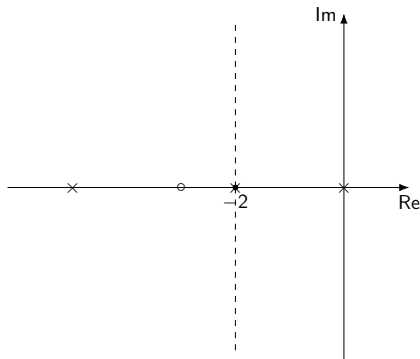


Reading the sketch. Three branches start at the three poles and eventually align with the dashed rays as $K \rightarrow \infty$.

Worked Example — Mixed Poles/Zeros Change the Centroid

$$L_0(s) = \frac{s+3}{s(s+2)(s+5)}: p = \{0, -2, -5\}, z = \{-3\} \Rightarrow n - m = 2.$$

$$\sigma_a = \frac{(0 - 2 - 5) - (-3)}{2} = \frac{-4}{2} = -2, \quad \theta_q = \frac{(2q+1)\pi}{2} \Rightarrow \{90^\circ, 270^\circ\}.$$



Interpretation. The finite zero pulls the centroid to the right; the asymptotes are vertical, so complex branches move up/down near $\text{Re}(s) = -2$.

Worked Micro-Example (Angle/Magnitude at a Trial Point)

Revisit $G(s) = \frac{1}{s(s+2)}$. Suppose you *guess* $s^* = -1 + j1$.

1. **Angle test.** Draw vectors from the poles at 0 and -2 to s^* ; sum their angles (with sign) and check if $\angle G(s^*) = 180^\circ \pmod{360^\circ}$. If not, adjust s^* until it is.
2. **Magnitude.** If the angle test passes, compute

$$K = \frac{1}{|G(s^*)|} = |s^*| |s^* + 2|.$$

Geometric note. For this plant the locus becomes vertical for $K > 1$ because the real part of the complex poles is fixed at -1 (from the quadratic formula).

Quick Practice (Real-Axis Segments Done Right)

A. $G(s) = \frac{s+3}{s(s+2)}$ (poles 0, -2; zero -3).

Intervals: $(-\infty, -3)$ **on**, $(-3, -2)$ **off**, $(-2, 0)$ **on**, $(0, \infty)$ **off**.

B. $G(s) = \frac{s+1}{(s+2)(s+4)}$ (poles -2, -4; zero -1).

Intervals: $(-\infty, -4)$ **on**, $(-4, -2)$ **off**, $(-2, -1)$ **on**, $(-1, \infty)$ **off**.

Compute a sample K : for case B at $s^* = -1.5$,

$$K = \frac{|s^* + 2| |s^* + 4|}{|s^* + 1|} = \frac{0.5 \cdot 2.5}{0.5} = 2.5.$$

Polling — Concept Check (with Guidance)

Q1. Derive $\angle G(s) = 180^\circ + 360^\circ k$ starting from $1 + KG(s) = 0$.

Q2. State and justify the real-axis rule succinctly.

Q3. For $G(s) = \frac{1}{s(s+1)(s+3)}$, which real-axis segments are on the RL? (Make the odd-right-count table.)

Q4. For $G(s) = \frac{s+2}{s(s+4)}$, test $s^* = -2 + j2$ with the angle rule — on or off?

Q5. Once the angle test succeeds at s^* , how is K obtained and why is this step second?

Answers — Concept Check (Concise)

- A1.** $1 + KG = 0 \Rightarrow KG = -1 = 1\angle(180^\circ + 360^\circ k)$; with $K > 0$, $\angle G = 180^\circ + 360^\circ k$ and $|G| = 1/K$.
- A2.** On the real axis, each real pole/zero to the right contributes 180° ; the sum is $180^\circ \pmod{360^\circ}$ iff the count is odd.
- A3.** Poles at $0, -1, -3$: $(-\infty, -3)$ on, $(-3, -1)$ off, $(-1, 0)$ on, $(0, \infty)$ off.
- A4.** For $s^* = -2 + j2$: zero angle = 90° , pole angles = 135° and 45° , so $\angle G = -90^\circ \not\equiv 180^\circ$; it is off the RL.
- A5.** $K = 1/|G(s^*)|$ (or product-distance formula). Angle first ensures the point lies on the locus; magnitude then reads the compatible K .

Section 7.3 Overview — The 7-Step Sketching Procedure

Objective: Sketch the closed-loop pole trajectories of $1 + KG(s) = 0$ as $K \geq 0$ varies.

Inputs: $G(s) = \frac{\prod_{i=1}^M (s - z_i)}{\prod_{j=1}^n (s - p_j)}$. **Outputs:** Locus geometry, crossings, break points, angles, and gain values.

Procedure (7 steps):

1. Plot open-loop poles ($\times$) and zeros ($\circ$).
2. Mark real-axis segments that belong to the locus.
3. Draw asymptotes to infinity (angles & centroid).
4. Compute breakaway/break-in points on real segments.
5. Find $j\omega$ -axis crossings (Routh–Hurwitz).
6. Determine departure/arrival angles at complex poles/zeros.
7. Read/compute the gain K at selected points; complete the sketch.

Step 1 — Pole/Zero Map and Branch Count

Write $G(s)$ in factored form:

$$G(s) = \frac{\prod_{i=1}^M (s - z_i)}{\prod_{j=1}^n (s - p_j)}, \quad n = \deg D, \quad M = \deg N.$$

Start/End of branches:

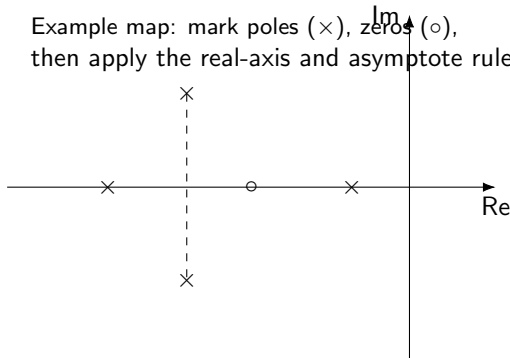
- Loci *start* at open-loop poles p_j as $K \rightarrow 0^+$.
- Loci *end* at open-loop zeros z_i as $K \rightarrow \infty$.
- If $n > M$, there are $N_\infty = n - M$ branches going to infinity.

Symmetry: Locus is symmetric about the real axis.

Action: Place all p_j ($\times$) and z_i ($\circ$) on the s -plane.

Step 1 — Pole/Zero Skeleton (Self-Contained TikZ)

Example map: mark poles ($\times$), zeros ($\circ$),
then apply the real-axis and asymptote rules.



Step 2 — Real-Axis Segments (Derivation & Rule, Expanded)

Setup. Let $G(s) = \frac{\prod_{i=1}^M (s - z_i)}{\prod_{j=1}^n (s - p_j)}$ and test a real point $s^* \in \mathbb{R}$.

Angle condition (unity-feedback, $K > 0$):

$$\angle G(s^*) = \sum_{i=1}^M \angle(s^* - z_i) - \sum_{j=1}^n \angle(s^* - p_j) \equiv 180^\circ \pmod{360^\circ}.$$

Key fact on the real line. For any real a ,

$$\angle(s^* - a) = \begin{cases} 0^\circ, & s^* > a \text{ (point is to the right of } a), \\ 180^\circ, & s^* < a \text{ (point is to the left of } a). \end{cases}$$

Therefore only *real* poles/zeros contribute nonzero angles, and they contribute either 0° or 180° .
Complex-conjugate pairs *not* on the real axis contribute angles that cancel in conjugate-symmetric fashion.

Define $N_{\text{right}}(s^*) = \#$ of real poles+zeros strictly to the *right* of s^* . Then

$$\angle G(s^*) = 180^\circ \times N_{\text{right}}(s^*) \pmod{360^\circ}.$$

Conclusion (for $K > 0$). $\angle G(s^*) \equiv 180^\circ \pmod{360^\circ}$

$$\iff N_{\text{right}}(s^*) \text{ is } \mathbf{odd}.$$

Rule: A real point s^* lies on the root locus iff the number of real poles and zeros to its right is **odd**. □

Note (negative-gain locus). If you sketch for $K < 0$ (rare but useful), the angle condition becomes $0^\circ \pmod{360^\circ} \Rightarrow$ *even* count.

Step 2 — Practical Checklist, Edge Cases, and Mini-Example

How to apply on paper (fast):

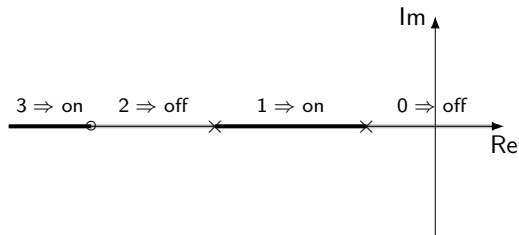
1. Sort all *real* poles ($\times$) and zeros ($\circ$) on the real axis.
2. Partition the real line into intervals between consecutive marks.
3. For a test point in each interval, count real marks to its *right*.
4. If the count is **odd** ($K > 0$ case), the entire interval is on the locus; if **even**, it is off.

Edge cases to remember.

- At a marked location (s^* exactly equals a pole/zero): the angle is undefined for the simple test. Poles are *start points* of branches; multiple poles/zeros (multiplicity > 1) often signal breakaway/break-in nearby (handled in later steps).
- Complex poles/zeros not on the real axis do *not* affect Step 2: they do not change parity along the real line.
- Do not mix signs: this rule assumes the standard positive- K locus (the usual negative-feedback characteristic equation $1 + KG(s) = 0$).

Mini-example. Real marks at -5 ($\circ$), -3.2 ($\times$), -1 ($\times$). Intervals:

$$\begin{array}{ll} (-\infty, -5) : 3 \text{ (odd)} \Rightarrow \text{ON} & (-5, -3.2) : 2 \text{ (even)} \Rightarrow \text{OFF} \\ (-3.2, -1) : 1 \text{ (odd)} \Rightarrow \text{ON} & (-1, \infty) : 0 \text{ (even)} \Rightarrow \text{OFF} \end{array}$$



Step 3 — Asymptotes to Infinity (Angles)

When $n > M$, $N_\infty = n - M$ branches go to infinity along straight-line **asymptotes**.

Angles:

$$\phi_k = \frac{(2k + 1)180^\circ}{n - M}, \quad k = 0, 1, \dots, N_\infty - 1.$$

Interpretation: rays centered on the real axis that spread uniformly.

Note: These angles do not depend on pole/zero locations individually—only on their counts.

Step 3 — Asymptote Centroid (Real Intercept)

The **centroid** (real intercept) is

$$\sigma_A = \frac{\sum_{j=1}^n p_j - \sum_{i=1}^M z_i}{n - M}$$

where p_j, z_i denote all pole/zero locations; conjugate pairs have real sums and therefore contribute twice their real parts.

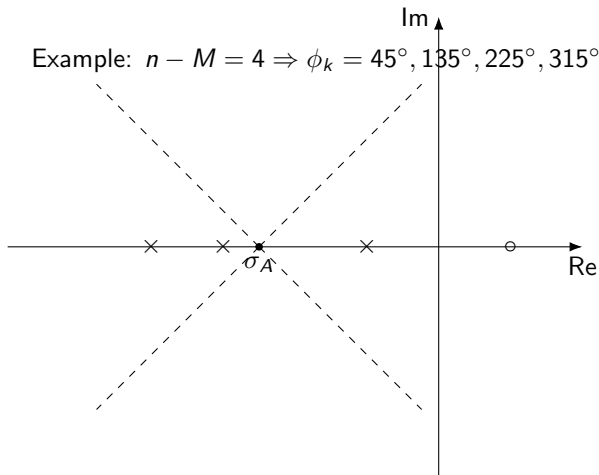
Construction:

1. Compute σ_A .
2. Draw rays from $(\sigma_A, 0)$ at angles ϕ_k .

Step 3 — TikZ Asymptote Sketch (Placeholder)

Replace data by your case or insert textbook figure here.

Example: $n - M = 4 \Rightarrow \phi_k = 45^\circ, 135^\circ, 225^\circ, 315^\circ$.



Step 4 — Breakaway/Break-in Points (Derivation)

Rewrite CE as $1 + KG(s) = 0 \Rightarrow K = -\frac{1}{G(s)}$. On a real-axis locus segment,

$$K(s) = -\frac{1}{G(s)} \quad (\text{on a real-axis locus segment, this is positive because the angle rule is satisfied}).$$

Break point condition (local extremum of K along the real segment):

$$\boxed{\frac{dK}{ds} = 0}.$$

Operational form: Solve $\frac{dK}{ds} = 0$ where $K = \frac{-1}{G(s)}$ or equivalently

$$\frac{d}{ds} \left[-\frac{1}{G(s)} \right] = 0 \quad \Longleftrightarrow \quad \frac{G'(s)}{G(s)^2} = 0 \quad \text{where } G(s) \neq 0, \infty.$$

Practical trick: Express K directly from CE written as a polynomial: $K = f(s)$, then solve $\frac{df}{ds} = 0$ on RL segments.

Step 4 — Break Point Example (Real Segment)

Example: $G(s) = \frac{1}{s(s+2)} \Rightarrow \Delta(s) = 1 + \frac{K}{s(s+2)} = 0 \Rightarrow s^2 + 2s + K = 0.$

Solve for K :

$$K(s) = -s(s+2) = -s^2 - 2s.$$

Break point(s) on real RL segments satisfy

$$\frac{dK}{ds} = -2s - 2 = 0 \Rightarrow s = -1.$$

Check $s = -1$ lies on a real-axis RL segment (yes: between -2 and 0 is *on* by the odd-count rule). The two branches coalesce there before departing into the complex plane, consistent with $s = -1 \pm \sqrt{1-K}$ giving a double root at $K = 1$.

Reminder: Break points must lie on RL; verify with the angle rule.

Step 5 — Imaginary-Axis Crossings (Routh–Hurwitz)

Idea: Determine K for which the CE has roots on $s = \pm j\omega$.

Procedure:

1. Multiply CE to a polynomial in s : $\Delta(s) = a_0 + a_1s + \cdots + a_ns^n = 0$ (coefficients depend on K).
2. Construct the Routh array; require all first-column entries positive for strict stability; boundary cases occur when a first-column entry becomes zero or a zero row appears.
3. The *borderline* (a zero row or sign change) yields values of K and the auxiliary equation.
4. Solve the auxiliary equation to get ω (the crossing point(s) $s = \pm j\omega$).

Note: Mark crossings on the $j\omega$ axis and indicate direction of motion as K increases (via angle rule).

Step 5 — Mini Example (Outline)

Given: $\Delta(s) = s^3 + a_2(K)s^2 + a_1(K)s + a_0(K) = 0$.

Routh array:

$$\begin{array}{c|cc} s^3 & 1 & a_1(K) \\ s^2 & a_2(K) & a_0(K) \\ s^1 & \frac{a_2(K)a_1(K) - a_0(K)}{a_2(K)} & 0 \\ s^0 & a_0(K) & \end{array}$$

Crossing: set the s^1 entry $= 0 \Rightarrow a_2a_1 - a_0 = 0$ to get candidate K . Substitute into auxiliary equation $s^2 + a_0/a_2 = 0$ to find ω .

Action: Place $s = \pm j\omega$ and record the corresponding K .

Step 6 — Departure and Arrival Angles (Complex Points)

Departure angle at a complex pole p_k :

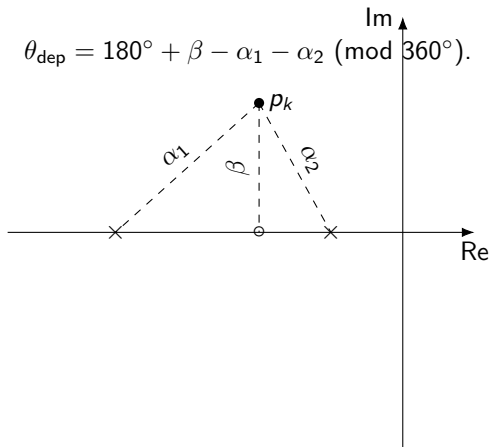
$$\theta_{\text{dep}}(p_k) = 180^\circ + 360^\circ m + \sum_{i=1}^M \angle(p_k - z_i) - \sum_{\substack{j=1 \\ j \neq k}}^n \angle(p_k - p_j) .$$

Arrival angle at a complex zero z_ℓ :

$$\theta_{\text{arr}}(z_\ell) = 180^\circ + 360^\circ m - \left[\sum_{\substack{i=1 \\ i \neq \ell}}^M \angle(z_\ell - z_i) - \sum_{j=1}^n \angle(z_\ell - p_j) \right] .$$

Note: Use principal values and preserve conjugate symmetry. For real poles/zeros, the branch direction along the real axis is determined by the real-axis segments adjacent to that point.

Step 6 — Angle Picture (TikZ aide)



Step 7 — Gain Reading & Completing the Sketch

At a candidate point s^* on RL:

$$\text{Angle test: } \sum_i \angle(s^* - z_i) - \sum_j \angle(s^* - p_j) \equiv 180^\circ \pmod{360^\circ}.$$

If true,

$$K = \frac{\prod_{j=1}^n |s^* - p_j|}{\prod_{i=1}^M |s^* - z_i|}.$$

Complete the sketch by:

- Connecting real-axis segments, asymptotes, and complex departures smoothly (respect symmetry).
- Marking break points and $j\omega$ crossings.
- Indicating direction of motion for increasing K (from poles toward zeros).

Consolidated Mini-Example (Walkthrough)

Plant: $G(s) = \frac{s+3}{s(s+2)(s+4)}$ ($n = 3$, $M = 1$, $N_\infty = 2$).

1) Poles/zeros: $p = \{0, -2, -4\}$, $z = \{-3\}$.

2) Real-axis RL: $(-\infty, -4)$: off; $(-4, -3)$: on; $(-3, -2)$: off; $(-2, 0)$: on; $(0, \infty)$: off.

3) Asymptotes: $\phi_k = \frac{(2k+1)180^\circ}{2} = \{90^\circ, 270^\circ\}$; $\sigma_A = \frac{(0-2-4) - (-3)}{2} = \frac{-3}{2} = -1.5$.

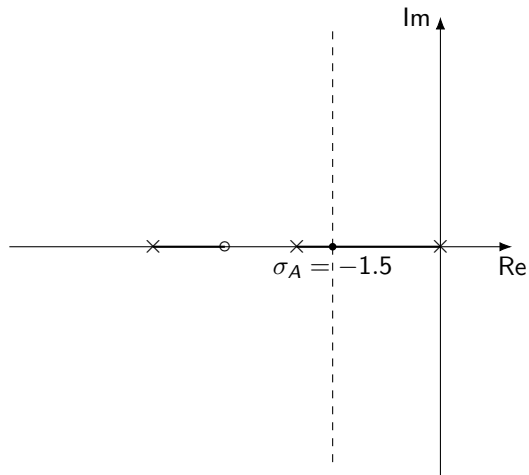
4) Break points: Write $K = f(s)$ from CE and solve $\frac{df}{ds} = 0$ on the on-RL intervals.

5) $j\omega$ crossings: Build Routh table of the CE polynomial to find K and ω .

6) Departure/arrival: At complex sections, use the formulas to get angles.

7) Gain read: At any s^* on RL, compute K by the magnitude rule.

Consolidated Mini-Example — Sketch Scaffold (TikZ)



Use as a scaffold; refine with your computed points.

Checklist & Common Pitfalls

- Confirm n, M : wrong counts $\Rightarrow$ wrong asymptotes/centroid.
- Real-axis rule applies only on the real line and is necessary (not sufficient) for full geometry.
- Break points must lie on RL segments (verify angle rule after solving $\frac{dK}{ds} = 0$).
- Use Routh to place exact $j\omega$ crossings; do not guess from sketches.
- Maintain symmetry; keep branch continuity; indicate direction as K increases.

Q1. For $G(s) = \frac{1}{s(s+4)(s+6)}$, find n , M , N_∞ and the asymptote angles ϕ_k .

Q2. For $G(s) = \frac{s+2}{s(s+3)}$, mark which real-axis intervals are on the RL.

Q3. With $G(s) = \frac{s+1}{s(s+2)(s+5)}$, compute the asymptote centroid σ_A .

Q4. Outline how you would compute a breakaway point on $(-5, -2)$ if it is on-RL.

Q5. For a cubic CE $\Delta(s) = s^3 + a_2(K)s^2 + a_1(K)s + a_0(K)$, which Routh entry vanishing typically signals a $j\omega$ crossing? What auxiliary equation do you solve next?

A1. $n = 3, M = 0 \Rightarrow N_{\infty} = 3; \phi_k = \frac{(2k+1)180^\circ}{3} = \{60^\circ, 180^\circ, 300^\circ\}.$

A2. Poles at $0, -3$; zero at -2 . Intervals: $(-\infty, -3)$ on; $(-3, -2)$ off; $(-2, 0)$ on; $(0, \infty)$ off.

A3. $\sigma_A = \frac{(0 - 2 - 5) - (-1)}{3 - 1} = \frac{-6 + 1}{2} = -2.5.$

A4. Write CE polynomial $\Delta(s) = 0 \Rightarrow K = f(s)$ on the real segment; solve $\frac{df}{ds} = 0$; keep only roots lying on the on-RL portion and satisfying the angle rule.

A5. The s^1 -row entry typically becomes zero (for a cubic). Solve the *auxiliary* equation formed from the row above (the s^2 row) to get ω .

Design by Root Locus — Idea & Workflow (Step-by-Step)

Goal (what you design): choose controller/plant parameters so the *closed-loop poles* fall inside a desired region that encodes damping ζ , speed (via T_s or σ), and sometimes bandwidth or ω_n .

Core equation: characteristic equation (CE) $1 + L(s, \theta) = 0$, with tunables θ (gains, zero/pole locations).

Design workflow you can repeat:

1. **Expose a single scalar** α in the CE: rewrite $1 + \alpha F(s) = 0$ so that the root locus is drawn vs. $\alpha \geq 0$.
2. **Sketch the root locus** of $F(s)$ quickly: poles/zeros, real-axis segments, asymptotes, breakaway hints.
3. **Overlay performance region** in the s -plane (lines of constant ζ and σ).
4. **Pick a target point** s^* *on the locus* inside the region; compute the required $\alpha^* = \frac{1}{|F(s^*)|}$.
5. **Back-solve** to physical parameters (e.g., K_P, K_I, K_D) from α^* ; check steady-state error type.
6. **Validate & iterate:** verify with time responses; if non-dominant poles spoil specs, slightly move s^* or add a zero/adjust weights.

Mental model: geometry (angle rule) *selects* viable s^* ; arithmetic (magnitude rule) *reads* the parameter value that lands you there.

Map Time-Domain Specs to the s -Plane (with Numbers)

Damping lines. For a 2nd-order dominant pair $s = \sigma \pm j\omega_d$,

$$\zeta = \frac{-\sigma}{\sqrt{\sigma^2 + \omega_d^2}} = \cos \theta \Rightarrow \theta = \cos^{-1} \zeta.$$

A damping constraint $\zeta \geq \zeta_{\min}$ is a sector to the *left* of the ray at angle $\theta_{\min}$ from the origin.

Speed via settling time. Using the 2% rule, $T_s \approx \frac{4}{|\sigma|}$ with $\sigma = \text{Re}\{s\} < 0$.

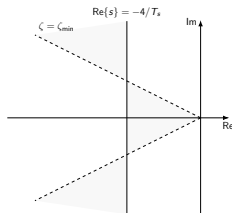
Demand $T_s \leq T_s^* \Rightarrow$ vertical line $\text{Re}\{s\} = -\sigma^*$ where $\sigma^* = \frac{4}{T_s^*}$; feasible region is *to the left*.

Overshoot/Peak time (dominant 2nd-order approx).

$$M_p \approx e^{-\pi\zeta/\sqrt{1-\zeta^2}}, \quad \omega_d = \omega_n \sqrt{1-\zeta^2}, \quad T_p = \frac{\pi}{\omega_d}.$$

Example mapping: “ $M_p \leq 10\%$ and $T_s \leq 1.5$ s”

$M_p \leq 0.10 \Rightarrow \zeta \approx 0.59$ (lookup or invert formula); $T_s \leq 1.5 \Rightarrow \sigma^* = 4/1.5 \approx 2.67$, i.e., $\text{Re}\{s\} \leq -2.67$.



Design region = intersection of (to the left of the σ -line) & (inside the damping wedge).

Expose a Parameter as a Gain (Make an RL in α)

If you already have a pure gain: $L(s, \theta) = K G(s)$ with $K > 0$,

$$1 + KG(s) = 0 \quad \Rightarrow \quad K = \frac{1}{|G(s^*)|} \text{ at any locus point } s^*.$$

If your knob is not a pure gain: rearrange the CE so the knob multiplies a transfer function.

$$\Delta(s, \alpha) = a_0(s) + \alpha a_1(s) = 0 \iff 1 + \alpha \frac{a_1(s)}{a_0(s)} = 0 \quad (\text{treat } \alpha \text{ as the "gain"}).$$

What can be α ? plant gain, sensor gain, K_P with other terms frozen, a scheduled pole/zero location after algebra, etc.

Checklist to expose α :

1. Push all fixed terms to one side; factor the *tunable* parameter α out.
2. Divide by the remaining polynomial to get $1 + \alpha F(s) = 0$.
3. Draw the RL of $F(s)$ versus $\alpha \geq 0$.

Example — Proportional Gain K_P as the RL Parameter

Unity feedback, $G(s) = \frac{b_0}{s(s+a)}$, controller $C(s) = K_P$.

$$L(s) = K_P G(s), \quad 1 + \frac{K_P b_0}{s(s+a)} = 0 \iff s^2 + as + K_P b_0 = 0.$$

How to design:

1. Sketch RL for $K_P \geq 0$ (poles at 0 and $-a$, no finite zero).
2. Overlay ζ -ray and σ -line; choose s^* on the locus inside the region.
3. Read K_P via magnitude:

$$K_P = \frac{1}{|G(s^*)|} = \frac{|s^*| |s^* + a|}{b_0}.$$

Steady-state note. With $C(s) = K_P$ and this plant having one integrator, the unity-feedback loop is type 1: step error is zero and ramp error is finite. Add integral action only when a higher type is required, and then recheck stability.

Two-Parameter Design — A Reliable Sequential Playbook

Consider $C(s; \alpha, \beta)$ (e.g., PD with $\alpha = K_D$, $\beta = K_P$; $z = \beta/\alpha$ sets the zero location).

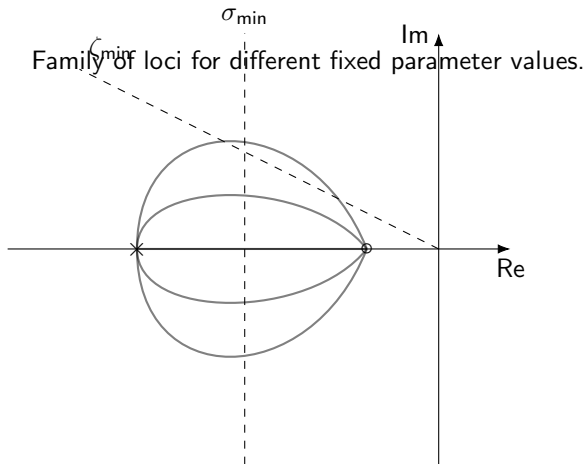
Sequential plan you can execute by hand:

1. **Fix the shape parameter** (for PD/lead, choose z), then treat the gain α as the RL parameter: draw $1 + \alpha F_z(s) = 0$.
2. Pick α^* so the locus passes a desired s^* (dominant poles in region).
3. **Freeze** $\alpha = \alpha^*$, then adjust the shape parameter slightly if steady-state or fine transient specs require another locus.

Why it works: each stage reduces the coupled design to a familiar 1D sweep, keeping geometry intuitive.

Root contours. To see coupling, plot several loci for multiple fixed values of one parameter and scan which pair (α, β) positions the poles best.

Root Contours — Visualizing the Tradeoffs



Use: overlay the design wedge; choose the curve (fixed parameter) whose locus crosses the wedge in a convenient place (reasonable gain, margin to $\text{Re} > 0$).

PD/Lead: Place a Zero First, Then Tune the Gain (with Numbers)

Let $C(s) = K_D s + K_P = K_D(s + z)$, $z = K_P/K_D > 0$.

Strategy intuition. The added zero bends the locus toward the damping wedge; then K_D selects *where* on that bent locus you land.

Example. Plant $G(s) = \frac{1}{s(s+4)}$. Specs: $\zeta \geq 0.6$, $T_s \leq 1.5$ s $\Rightarrow \operatorname{Re}\{s\} \leq -2.67$.

Choose a desired point $s^* = -2.8 \pm j3.73$ (about $\zeta = 0.6$ and left of -2.67). The angle rule then gives a feasible PD zero near -13.62 ; use $z = 13.62$.

$$K_D = \frac{1}{|(s^* + z)G(s^*)|} = \frac{|s^*||s^* + 4|}{|s^* + 13.62|} \quad (\text{since } G = 1/(s(s+4))).$$

Numerics (one conjugate):

$$|s^*| \approx 4.67, \quad |s^* + 4| \approx 3.92, \quad |s^* + 13.62| \approx 11.45,$$

$$K_D \approx \frac{4.67 \times 3.92}{11.45} \approx 1.60, \quad K_P = zK_D \approx 21.8.$$

Sanity checks: (i) does the closed-loop polynomial place the selected pair? (ii) are control effort and noise amplification acceptable? (iii) time response OK?

PI: Improve Steady-State Error, Keep Transients Tame

Take $C(s) = K_P + \frac{K_I}{s} = K_P \frac{s+z}{s}$ with $z = \frac{K_I}{K_P}$.

$L(s) = K_P \frac{s+z}{s} G(s) \Rightarrow$ draw RL in K_P with zero at $-z$ and extra pole at 0.

Design sequence:

1. Choose z to shape the locus toward the damping wedge (a moderate z gives a lead-like push).
2. With z fixed, pick s^* in the wedge *on the locus*.

3. Compute

$$K_P = \frac{1}{\left| \frac{s^* + z}{s^*} G(s^*) \right|}, \quad K_I = z K_P.$$

4. Verify type increase $\Rightarrow$ improved steady-state error; adjust z if transients overshoot or slow down.

Tip. If the added integrator makes a real-axis segment unstable (crossing to $\text{Re} > 0$ for some K_I), move z slightly or reduce K_I to avoid poor margins.

Back-Solving to Physical Parameters & Practical Constraints

Often K is a product: $K = k_{\text{amp}} k_{\text{sensor}} k_{\text{motor}} \cdots$.

Procedure you should follow:

1. From s^* , read K^* (or K_I^* , K_D^* , etc.) via the magnitude rule.
2. Solve for feasible hardware/software settings that achieve K^* *within limits* (saturation, quantization, rate limits).
3. If infeasible, slightly shift s^* within the region or tweak zero placement; recompute.

Good engineering hygiene.

- Prefer parameter changes that minimally amplify noise (avoid huge K_D near sensor noise).
- Keep a margin to the $\text{Re}\{s\} = 0$ axis; ensure gain/phase margins with a quick Bode/Nyquist check.
- Validate with step/disturbance simulations; measure M_p , T_s , IAE/ISE if relevant.

Worked Micro-Case — Algebraic Isolation (PD Example)

Unity feedback; $G(s) = \frac{b_0}{s(s+a)}$; $C(s) = K_D(s+z)$.

$$1 + K_D(s+z) \frac{b_0}{s(s+a)} = 0 \iff s(s+a) + K_D b_0(s+z) = 0.$$

Collect coefficients:

$$s^2 + (a + K_D b_0)s + K_D b_0 z = 0.$$

Dominant 2nd-order template: $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$ (for the *dominant* pair).

$$a + K_D b_0 = 2\zeta\omega_n, \quad K_D b_0 z = \omega_n^2.$$

Solve explicitly:

$$K_D = \frac{2\zeta\omega_n - a}{b_0}, \quad K_P = z K_D = \frac{z(2\zeta\omega_n - a)}{b_0}.$$

Caveat. This matches the dominant quadratic; check where the third pole goes in the full system. If additional plant/controller poles are present in the full system and are not far left, the responses may deviate from the simple template.

Plot quickly: `rlocus(L)` to visualize loci; `rlocfind` (or data cursor) to pick s^* and read the corresponding parameter.

Root contours: sweep one parameter over a grid of fixed values of the other to see how the locus family moves; overlay your performance wedge.

Checklist after plotting:

- Confirm s^* lies on the locus (angle rule). Read parameter (magnitude rule).
- Verify closed-loop stability for the chosen parameter (no poles in $\text{Re} > 0$).
- Simulate step/disturbance and verify M_p , T_s , steady-state error; adjust zeros/gains as needed.

Remember: RL gives geometry and a good first pick; always validate with time-domain simulations and margins.

- Q1.** How do you convert a non-gain parameter α in the CE into a root-locus “gain” form? Show the algebraic step.
- Q2.** Explain how ζ and T_s constraints are drawn in the s -plane and why their intersection forms a design region.
- Q3.** In a PD design $C(s) = K_D(s + z)$, why is it common to choose z first and then tune K_D ?
- Q4.** For $C(s) = K_P + \frac{K_I}{s}$, how would you carry out a sequential two-parameter design using RL?
- Q5.** Given a desired dominant pair $s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$, show how to back-solve (K_D, K_P) for the quadratic approximation in the PD example.

A1. If $\Delta(s, \alpha) = a_0(s) + \alpha a_1(s) = 0$, divide by $a_0(s)$: $1 + \alpha \frac{a_1(s)}{a_0(s)} = 0$. Treat α as the RL gain on $F(s) = a_1/a_0$.

A2. ζ constraint: rays from the origin at $\theta = \cos^{-1} \zeta$. T_s constraint: vertical line $\operatorname{Re}\{s\} = -4/T_s$. Their intersection yields acceptable pole locations meeting both transient requirements.

A3. The zero location z shapes the locus geometry (phase lead), steering branches toward the ζ -ray; once geometry is favorable, K_D selects a point on that locus to meet σ (speed).

A4. Fix $z = K_I/K_P$ to shape transients; with z fixed, treat K_P as the RL gain in $L(s) = K_P \frac{s+z}{s} G(s)$, pick s^* and compute K_P ; then set $K_I = zK_P$. Iterate z if steady-state/transient trade-offs require.

A5. Match $s^2 + (a + K_D b_0)s + K_D b_0 z$ to $s^2 + 2\zeta\omega_n s + \omega_n^2$ to get $K_D = \frac{2\zeta\omega_n - a}{b_0}$ and $K_P = zK_D$ (details shown on the “Micro-Case” slide).

- Root locus: where closed-loop poles move as K varies; use angle & magnitude conditions.
- Sketch with rules: starts/ends, real-axis segments, asymptotes, breakaway, departure/arrival.
- Use Routh to find imaginary-axis crossings and K ranges.
- Design by adding gain/lead/lag to meet damping, speed, and steady-state specs.

Dorf, R. C. & Bishop, R. H., *Modern Control Systems*, 14th Ed., Chapter 7.

Slides include placeholder images you can replace with textbook figures or suitable web images.