

Chapter 8 — Frequency Response Methods (Bode, Margins, Examples)

Dorf & Bishop, Modern Control Systems (14th ed.)

Feedback Control
Gang Chen

October 10, 2026

What you will be able to do

We will answer five questions in sequence.

- ① What happens when the input is a sinusoid?
- ② How do poles and zeros shape a Bode plot?
- ③ How can an experiment reveal an unknown plant?
- ④ What do bandwidth, resonance, and stability margins tell us?
- ⑤ How can a controller meet a specification, and how do we check it?

Each worked example ends with a numerical or graphical check.

Why examine one frequency at a time?

A system can track slow motion yet reject fast vibration.

A step combines many frequencies. A sine test isolates one frequency, so its effect is easier to interpret.

- Low frequencies: tracking and disturbances
- Mid frequencies: speed and damping
- High frequencies: noise and neglected dynamics

Notation used throughout

$P(s)$	plant transfer function
$C(s)$	controller transfer function
$u(t), y(t)$	plant input and output
s	Laplace variable
ω	angular frequency, rad/s
$j = \sqrt{-1}$	imaginary unit
$\omega = 2\pi f$, where f is in hertz.	

The sinusoidal steady-state response

A stable linear time-invariant system preserves sinusoidal frequency.

For input $u(t) = A \sin \omega t$ and transfer function $G(s)$,

$$y_{ss}(t) = A |G(j\omega)| \sin(\omega t + \phi(\omega)), \quad \phi(\omega) = \angle G(j\omega).$$

- $|G(j\omega)|$ is the output/input amplitude ratio.
- $\phi(\omega)$ is the phase shift: negative phase means lag.
- The transient must decay before this relation describes the total output.

The frequency response is the complex-valued function $G(j\omega)$.

Assumptions: stable LTI dynamics, a finite response at the test frequency, and no saturation.

Why substitution $s = j\omega$ works

A complex sinusoid is an eigenfunction of an LTI system.

Let $g(t)$ be the impulse response. For a steady complex input $u_c(t) = Ae^{j\omega t}$,

$$\begin{aligned}y_c(t) &= \int_0^\infty g(\tau) Ae^{j\omega(t-\tau)} d\tau \\&= Ae^{j\omega t} \underbrace{\int_0^\infty g(\tau) e^{-j\omega\tau} d\tau}_{G(j\omega)} \\&= A|G(j\omega)|e^{j(\omega t + \phi)}.\end{aligned}$$

Taking the imaginary part gives the sine-response formula on the previous page.

The transfer function sets amplitude and phase; it does not change frequency.

For a stable causal system, this integral is the Fourier transform of $g(t)$ and the Laplace transform evaluated on the imaginary axis.

Worked example: a first-order low-pass filter

Given

$$G(s) = 1/(1 + 0.5s)$$

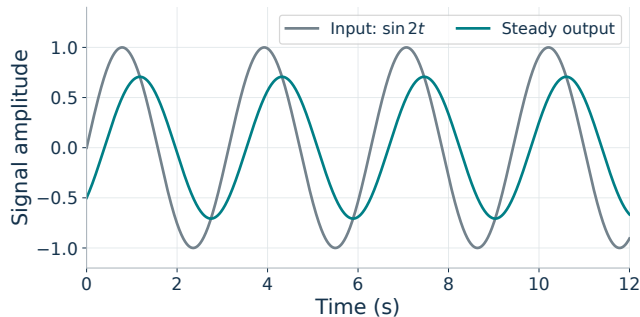
$$u(t) = \sin 2t$$

At $\omega = 2 \text{ rad/s}$,

$$G(j2) = \frac{1}{1+j} = \frac{1-j}{2}.$$

Thus $|G| = 1/\sqrt{2}$ and $\phi = -45^\circ$.

$$y_{ss} = 0.707 \sin(2t - 45^\circ)$$



Same period; smaller amplitude; the output lags by
 $\pi/8 = 0.393 \text{ s}$.

Three plots of the same complex response

At each frequency, write $G(j\omega) = r e^{j\phi}$.

Plot	Coordinates	Useful question
Bode	$20 \log_{10} r$ and ϕ versus $\log_{10} \omega$	Which frequencies are amplified or delayed?
Polar	$(r \cos \phi, r \sin \phi)$	Where is the complex response in the plane?
Nichols	$(\phi, 20 \log_{10} r)$	How does the loop imply closed-loop response?

For the previous filter at $\omega = 2$: Bode (-3.01 dB, -45°); polar (0.5 , -0.5); Nichols (-45° , -3.01 dB).

We start with Bode plots because individual factors add cleanly.

Decibels turn products into sums

Magnitude in decibels

$$M(\omega) = 20 \log_{10} |G(j\omega)|.$$

If $G = G_1 G_2$, then

$$M = M_1 + M_2, \quad \phi = \phi_1 + \phi_2.$$

A decade means $\omega \mapsto 10\omega$; an octave means $\omega \mapsto 2\omega$.

Useful amplitude ratios

$ G $	$20 \log_{10} G $
0.1	-20 dB
$1/\sqrt{2}$	-3.01 dB
1	0 dB
2	+6.02 dB
10	+20 dB

0 dB means unchanged amplitude, not zero output.

Factor the transfer function, then add the magnitude and phase contributions.

Normalize before drawing a Bode plot

Write each real pole or zero as $1 + s/\omega_b$.

The number ω_b is its break frequency, in rad/s.

$$\frac{100(s+2)}{s(s+10)} = \frac{100 \cdot 2(1+s/2)}{s \cdot 10(1+s/10)} = \frac{20(1+s/2)}{s(1+s/10)}.$$

- ① Extract the constant gain: $K = 20$.
- ② Identify the origin factor: $1/s$.
- ③ Identify the finite zero and pole: $\omega_z = 2$, $\omega_p = 10$.

Do not lose the constants while normalizing: they set the vertical position.

Gain and poles or zeros at the origin

Factor	Magnitude	Phase
$K > 0$	$20 \log_{10} K \text{ dB}$	0°
$1/s$	$-20 \log_{10} \omega \text{ dB}$	-90°
s	$+20 \log_{10} \omega \text{ dB}$	$+90^\circ$
$1/s^m$	$-20m \log_{10} \omega \text{ dB}$	$-90m^\circ$

For $10/s$: magnitude is 20 dB at 1 rad/s and 0 dB at 10 rad/s.

An integrator contributes -20 dB/decade at every frequency.

A negative gain also changes phase by 180° . Use a continuous phase convention.

A real pole: derive the magnitude

$$G(s) = 1/(1 + \tau s)$$

τ is the time constant; $\omega_b = 1/\tau$. Set $x = \omega\tau$:

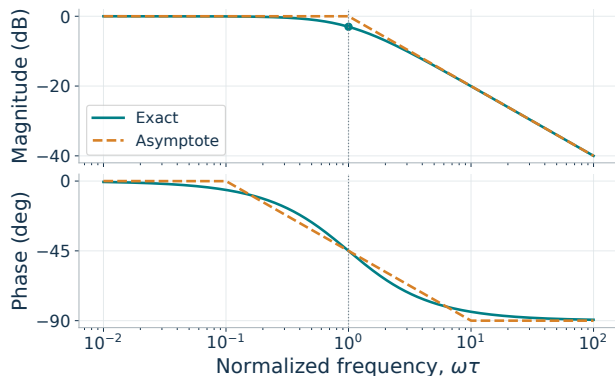
$$|G(j\omega)| = \frac{1}{\sqrt{1 + x^2}},$$

$$M = -10 \log_{10}(1 + x^2).$$

For $x \ll 1$: $M \approx 0$.

For $x \gg 1$: $M \approx -20 \log_{10} x$.

At $x = 1$: $M = -3.01$ dB.



A real pole: derive the phase

The denominator angle is subtracted because the pole is a reciprocal.

$$G(j\omega) = \frac{1}{1 + j\omega\tau}, \quad \phi(\omega) = -\tan^{-1}(\omega\tau).$$

$\omega\tau$	0.1	1	10
Exact phase	-5.71°	-45°	-84.29°
Straight-line sketch	0°	-45°	-90°

The usual sketch transitions over two decades, from $0.1/\tau$ to $10/\tau$.

Phase changes gradually around a break frequency, not abruptly at it.

A real zero reverses the pole contribution

For a left-half-plane zero, $G(s) = 1 + \tau s$.

$$M = +10 \log_{10}(1 + (\omega\tau)^2), \quad \phi = +\tan^{-1}(\omega\tau).$$

- Below $\omega_b = 1/\tau$: approximately 0 dB and 0° .
- At ω_b : +3.01 dB and $+45^\circ$.
- Above ω_b : the slope approaches +20 dB/decade and phase approaches $+90^\circ$.

A pole changes slope by -20 ; a zero changes it by $+20$ dB/decade.

Repeated factors multiply these changes by their multiplicity.

A second-order factor introduces damping

A complex pole pair needs two parameters, not just a break frequency.

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad \Omega = \frac{\omega}{\omega_n}.$$

Here ω_n is the natural frequency; ζ is the damping ratio.

$$G(j\omega) = \frac{1}{(1 - \Omega^2) + j2\zeta\Omega},$$
$$|G(j\omega)| = \frac{1}{\sqrt{(1 - \Omega^2)^2 + (2\zeta\Omega)^2}}.$$

At high frequency, two poles give a -40 dB/decade slope.

Second-order phase: keep the correct quadrant

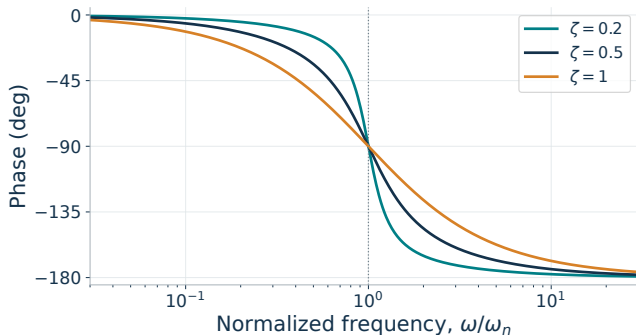
Use both parts: $\text{atan2}(b, a) = \angle(a + jb)$.

$$\phi = -\text{atan2}(2\zeta\Omega, 1 - \Omega^2).$$

- $\Omega \ll 1$: $\phi \rightarrow 0^\circ$
- $\Omega = 1$: $\phi = -90^\circ$
- $\Omega \gg 1$: $\phi \rightarrow -180^\circ$

A one-argument arctangent loses the quadrant when $1 - \Omega^2 < 0$.

Unwrap phase: add or subtract 360° to remove artificial jumps.



Resonance depends on damping

Minimize $D = (1 - \Omega^2)^2 + 4\zeta^2\Omega^2$. **Its derivative is** $D' = 4\Omega(\Omega^2 + 2\zeta^2 - 1)$.

The nonzero stationary point exists for $0 < \zeta < 1/\sqrt{2}$:

$$\omega_r = \omega_n \sqrt{1 - 2\zeta^2},$$

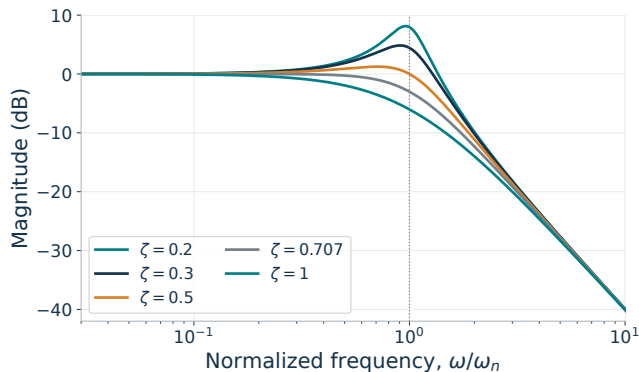
$$M_r = \frac{1}{2\zeta\sqrt{1 - \zeta^2}}.$$

At $\zeta = 0.3$:

$$\omega_r = 0.906\omega_n$$

$$M_r = 1.747, \text{ or } 4.85 \text{ dB.}$$

For $\zeta \geq 1/\sqrt{2}$, there is no peak above the DC value. M_r denotes a frequency-response peak, not step overshoot.



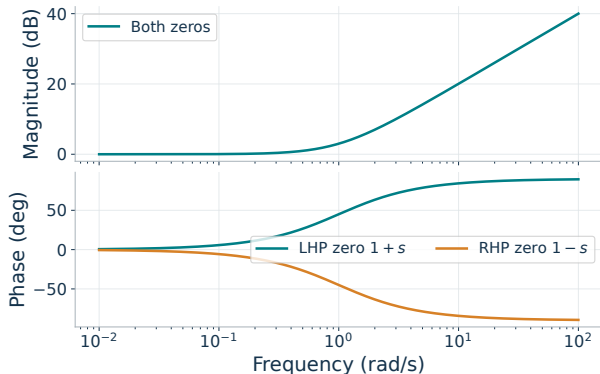
Equal magnitudes can hide different phases

Compare a left- and right-half-plane zero.

$G_1(s) = 1 + s$, $G_2(s) = 1 - s$. Both have magnitude $\sqrt{1 + \omega^2}$, but their phases have opposite signs.

A pure delay e^{-sT_d} adds another example:

$$|e^{-j\omega T_d}| = 1, \quad \phi = -\omega T_d \text{ rad.}$$



Magnitude alone cannot reveal a right-half-plane zero or a delay.

Worked Bode plot: identify the factors

We will sketch this model, then compare with the exact response.

$$G(s) = \frac{20(1 + s/5)}{s(1 + s/20)(1 + s/100)}.$$

Factor	Break (rad/s)	Slope contribution	Final phase
20	none	0	0°
1/s	origin	-20 everywhere	-90°
1 + s/5	5	+20 above break	+90°
(1 + s/20) ⁻¹	20	-20 above break	-90°
(1 + s/100) ⁻¹	100	-20 above break	-90°

Start at 1 rad/s: the low-frequency asymptote is 26.02 dB.

Worked Bode plot: add the slopes

Move left to right through the break frequencies.

Frequency range	dB/dec
$\omega < 5$	-20
$5 < \omega < 20$	0
$20 < \omega < 100$	-20
$\omega > 100$	-40

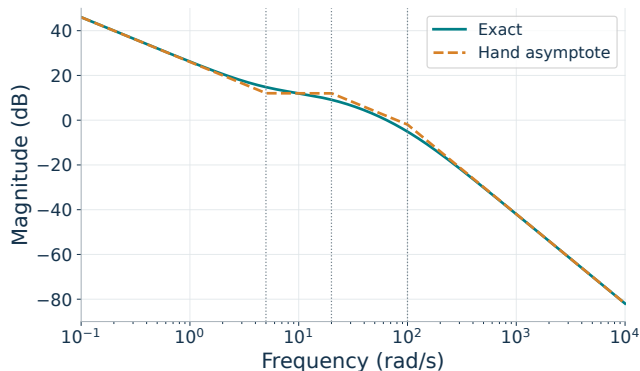
Asymptotic levels:

at 5: 12.04 dB

at 20: 12.04 dB

at 100: -1.94 dB

The corner changes the slope; it does not make a jump in magnitude.



Worked Bode plot: add the phases

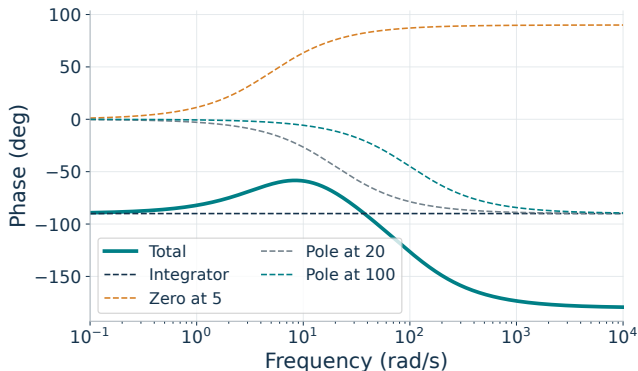
Add exact factor phases at the same frequency.

$$\begin{aligned}\phi(\omega) = & -90^\circ + \tan^{-1}(\omega/5) \\ & - \tan^{-1}(\omega/20) \\ & - \tan^{-1}(\omega/100).\end{aligned}$$

At 20 rad/s:

$$\phi = -90 + 75.96 - 45 - 11.31$$

$$\boxed{\phi = -70.35^\circ}.$$



The total phase is not obtained by reading only the nearest corner.

Check your understanding: a two-pole model

For $G(s) = 10/[(1 + 0.1s)(1 + 0.01s)]$, **find the response at 100 rad/s.**

① Breaks: 10 and 100 rad/s. Beyond both, the slope is -40 dB/decade.

② Asymptote at 100: $20 - 20 \log_{10}(100/10) = 0$ dB.

③ Exact magnitude:

$$20 - 10 \log_{10}(101) - 10 \log_{10}(2) = \boxed{-3.05 \text{ dB}}.$$

④ Exact phase: $-\tan^{-1}(10) - \tan^{-1}(1) = \boxed{-129.29^\circ}$.

A corner correction belongs to its own factor; do not count every pole as -3 dB.

Polar plots retain the complex geometry

Plot the real and imaginary parts as frequency increases.

For $G(j\omega) = a(\omega) + jb(\omega)$,

$$|G| = \sqrt{a^2 + b^2}, \quad \phi = \text{atan2}(b, a).$$

- 1 Find the low- and high-frequency limits.
- 2 Calculate a few points and any real- or imaginary-axis crossings.
- 3 Draw arrows to show increasing ω .

Radius gives magnitude; angle gives phase; frequency is a parameter along the curve.

For real-coefficient models, $G(-j\omega) = G(j\omega)^*$. The negative-frequency branch is a reflection about the real axis.

The first-order polar plot is a semicircle

Set $x = \omega\tau$.

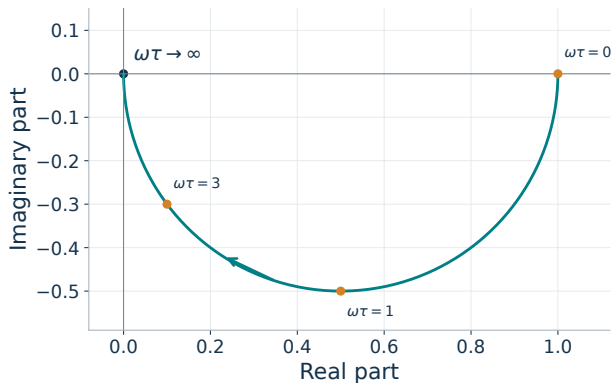
$$G(j\omega) = \frac{1 - jx}{1 + x^2} = a + jb.$$

Hence

$$a = \frac{1}{1 + x^2}, \quad b = -\frac{x}{1 + x^2}.$$

Eliminating x gives

$$\left(a - \frac{1}{2}\right)^2 + b^2 = \frac{1}{4}.$$



The positive-frequency branch moves from (1,0) to (0,0) below the real axis.

Polar geometry and the point -1

Feedback makes one location especially important.

Let R be the reference transform and $L = CP$. Feedback gives $Y = L(R - Y)$, so $Y/R = L/(1 + L)$. Its denominator is $1 + L$.

$$L(j\omega) = -1 \implies 1 + L(j\omega) = 0.$$

This means a closed-loop pole can lie on the imaginary axis.

- Near -1 , the denominator becomes small and closed-loop amplification can be large.
- A positive-frequency polar branch alone is not a complete stability test.
- A Nyquist test also accounts for the full contour and open-loop unstable poles.

Later, gain and phase margins will measure particular approaches to -1 .

Measure amplitude and phase experimentally

A sine test gives one point on the frequency response.

After transients decay, measure input amplitude A_u , output amplitude A_y , and time lag Δt :

$$|\hat{P}(j\omega)| = \frac{A_y}{A_u}, \quad \phi = -\omega\Delta t \quad (\text{radians, modulo } 2\pi).$$

- 1 Excite the plant gently at a selected ω .
- 2 Record input and output on the same time base.
- 3 Repeat at logarithmically spaced frequencies.

Example: $A_u = 2$, $A_y = 1$, $\omega = 4$, $\Delta t = 0.2$ s gives -6.02 dB and -45.84° .

An open-loop experiment must be safe. Stability alone does not guarantee acceptable motion or actuator effort.

Choose excitation and sampling deliberately

Excitation	What it provides	What to watch
Stepped sine	High precision at chosen frequencies	Slow; wait for transients
Chirp	Broad band in one sweep	Too-fast sweep blurs dynamics
Multisine	Many selected frequencies at once	Peak amplitude and periodic record
Random / PRBS*	Broad excitation spectrum	Averaging, leakage, actuator limits

*PRBS: pseudorandom binary sequence. For sample rate f_s and record duration T_{rec} :

$$f_{\text{Nyquist}} = f_s/2, \quad \Delta f = 1/T_{\text{rec}}.$$

Use anti-alias filtering and enough record length to resolve the slowest dynamics.

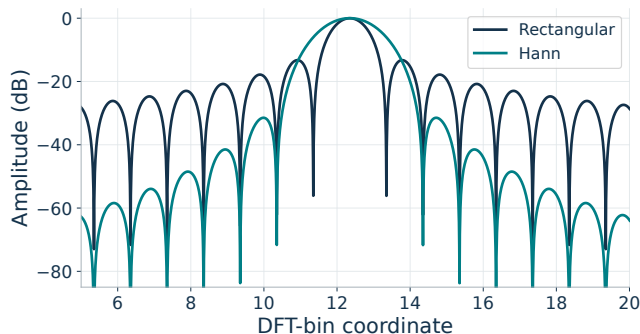
Finite records create spectral leakage

Why energy spreads into nearby frequency bins

The discrete Fourier transform (DFT) repeats the finite record. A noninteger number of tone cycles creates an endpoint jump.

- Use an integer number of cycles when possible.
- A Hann window lowers sidelobes but widens the main lobe.

Remove transients first. Averaging reduces variation, not leakage or bias.



Estimate a frequency response from spectra

At each DFT frequency sample (bin), average over K records.

Noise-free $Y = PU$ gives $S_{yu} = PS_{uu}$ and $S_{yy} = PS_{uy}$. Solve either for P to estimate the frequency response function (FRF). Let U_k, Y_k be DFT values; $*$ is conjugation.

$$S_{uu} = \langle U_k U_k^* \rangle, \quad S_{yy} = \langle Y_k Y_k^* \rangle, \quad S_{yu} = \langle Y_k U_k^* \rangle, \quad S_{uy} = S_{yu}^*.$$

$\langle \cdot \rangle$ denotes averaging over K records, with a common normalization.

$$\boxed{\hat{P}_{H_1} = \frac{S_{yu}}{S_{uu}}}, \quad \boxed{\hat{P}_{H_2} = \frac{S_{yy}}{S_{uy}}}.$$

The order of the cross-spectrum subscripts matters: reversing it conjugates the phase.

The same sample rate and window normalization must be used for all spectra.

Noise assumptions determine estimator bias

Suppose $Y = PU + N$ and N is uncorrelated with U .

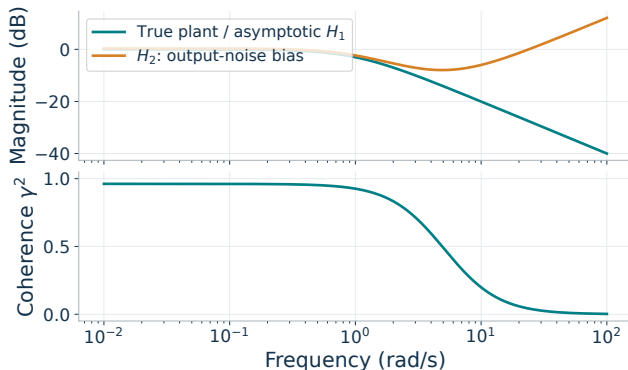
Then $S_{yu} = PS_{uu}$, so H_1 is asymptotically unbiased under this output-noise model.

Coherence is

$$\gamma^2 = \frac{|S_{yu}|^2}{S_{yy}S_{uu}} \in [0, 1].$$

Low coherence can indicate noise, weak excitation, leakage, or nonlinearity.

H_1 is biased low by independent input measurement noise; H_2 is biased high by independent output noise. High coherence is useful evidence, not proof of a correct experiment.



Worked spectral estimate

At one frequency, suppose $S_{uu} = 4$, $S_{yy} = 3$, and $S_{yu} = 2 - 2j$.

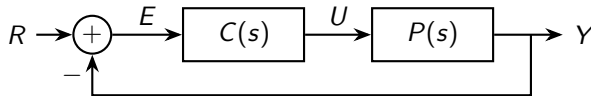
$$\begin{aligned}\hat{P}_{H_1} &= \frac{2 - 2j}{4} = 0.5 - 0.5j, \\ |\hat{P}_{H_1}| &= 0.707 \Rightarrow -3.01 \text{ dB}, \quad \phi = -45^\circ, \\ \gamma^2 &= \frac{|2 - 2j|^2}{3 \cdot 4} = \frac{8}{12} = 0.667, \\ \hat{P}_{H_2} &= \frac{3}{2 + 2j} = 0.75 - 0.75j.\end{aligned}$$

Under the output-noise model, the larger H_2 magnitude is expected.

Report the estimate together with coherence, excitation level, and repeatability.

Three transfer functions describe feedback

In the Laplace domain: reference R , error E , plant input U , and output Y .



$E = R - Y$, $U = CE$, and $Y = PU$ give $Y = PC(R - Y)$. Define $L = CP$:

$$\boxed{T = \frac{Y}{R} = \frac{L}{1 + L}}, \quad \boxed{S = \frac{E}{R} = \frac{1}{1 + L}}, \quad S + T = 1.$$

L : loop shape. T : reference tracking. S : remaining error.

Use L for loop margins; use T for tracking bandwidth and resonance.

Bandwidth is a closed-loop speed measure

For a low-pass T , define $M_T = 20 \log_{10} |T|$ and use a drop from the DC gain.

$$|T(j\omega_B)| = \frac{|T(0)|}{\sqrt{2}} \iff M_T(\omega_B) = M_T(0) - 3.01 \text{ dB}.$$

For $T(s) = 1/(1 + \tau s)$,

$$\omega_B = 1/\tau, \quad y(t) = 1 - e^{-t/\tau}, \quad t_{10-90} = \tau \ln 9 \approx \frac{2.20}{\omega_B}.$$

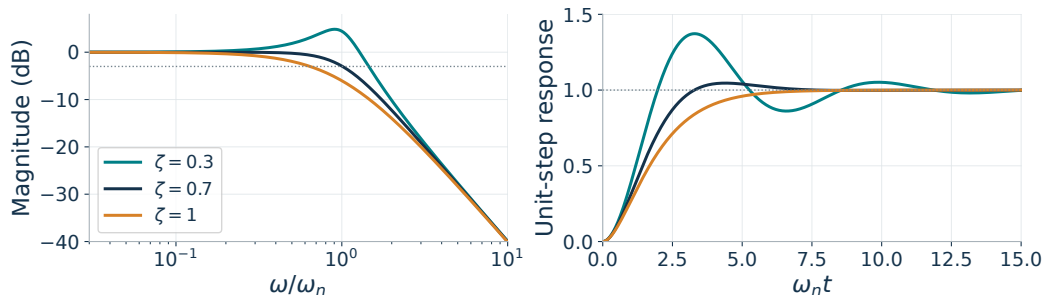
- Larger bandwidth usually means faster response for similar shapes.
- Gain crossover satisfies $|L| = 1$; it is not the definition of bandwidth.

The bandwidth threshold is -3 dB absolute only when $T(0) = 1$.

Resonance and step overshoot

For the standard second-order model, damping controls both responses.

For $0 < \zeta < 1$, step overshoot is $\%OS = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}}$.



These relations are exact for the standard model. Extra poles, zeros, or delays can invalidate a second-order approximation.

Gain and phase margins

Gain crossover ω_{gc}

At $|L(j\omega_{gc})| = 1$,

$$\text{PM} = 180^\circ + \angle L(j\omega_{gc}).$$

It is the extra phase lag needed to reach -180° at unit magnitude.

Phase crossover ω_{pc}

At $\angle L(j\omega_{pc}) = -180^\circ$,

$$\text{GM} = \frac{1}{|L(j\omega_{pc})|},$$

$$\text{GM}_{\text{dB}} = -20 \log_{10} |L(j\omega_{pc})|.$$

Both margins describe specific distances to the critical point -1 .

Use continuous phase and inspect all crossings. Positive scalar margins alone do not certify every loop, especially with unstable open-loop poles or multiple crossings.

Worked margins: one loop, two crossings

$$L(s) = 10/[s(1 + s/10)(1 + s/50)]$$

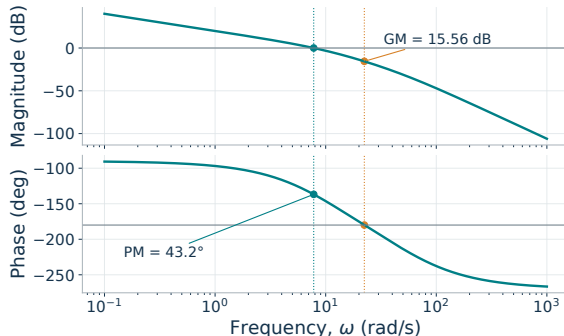
Gain crossover: $|L|^2 = 1$ gives

$$\omega^2(1 + \omega^2/100)(1 + \omega^2/2500) = 100.$$

Numerically, $\omega_{gc} = 7.793$ rad/s.

$$\begin{aligned} \text{PM} &= 90^\circ - \tan^{-1}(0.7793) \\ &\quad - \tan^{-1}(0.1559) = 43.21^\circ. \end{aligned}$$

Phase crossover: the pole angles sum to 90° , so $\omega^2 = 500$. Thus $\omega_{pc} = 22.36$ rad/s. There $|L| = 1/6$, so $\text{GM} = 6$ (15.56 dB).



Gain and delay change the margin differently

Use the previous loop as a reference.

Multiplying L by a positive gain k adds $20 \log_{10} k$ dB but no phase.

- The phase crossover stays at 22.36 rad/s.
- Increasing gain to $k = 6$ brings the loop exactly to -1 there.
- The gain crossover moves, so phase margin must be recomputed.

A pure delay adds $-\omega T_d$ radians of phase without changing magnitude. For a single relevant gain crossover,

$$T_{d,\text{margin}} = \frac{\text{PM}(\pi/180)}{\omega_{gc}} \approx \frac{0.754}{7.793} = 0.097 \text{ s.}$$

A faster loop tolerates less delay for the same phase margin.

Sensitivity, disturbances, and sensor noise

Add plant-input disturbance d and sensor noise n .

With feedback measurement $Y + N$ in the Laplace domain,

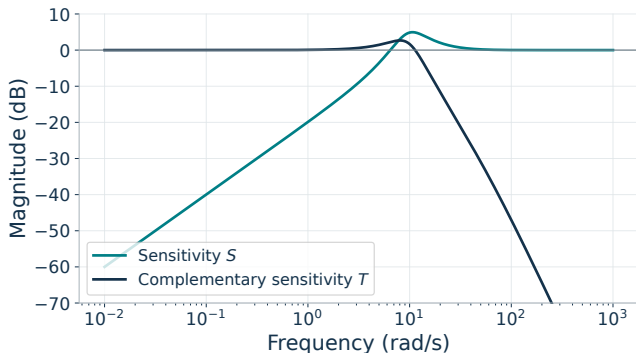
$$Y = P\{C[R - (Y + N)] + D\}.$$

Collecting Y terms gives

$$Y = TR + PSD - TN.$$

Large low-frequency $|L|$ makes $|S|$ small.
Small high-frequency $|L|$ usually makes $|T|$ small.

$S + T = 1$ is a complex identity. It does not imply $|S| + |T| = 1$ or force T to have a resonant peak.



Nichols coordinates combine magnitude and phase

Each frequency gives one point (ϕ, M) instead of two vertical readings.

$$\phi = \angle L(j\omega), \quad M = 20 \log_{10} |L(j\omega)|.$$

For example, if $L(j\omega) = 0.5e^{-j120^\circ}$:

- Bode magnitude: -6.02 dB; Bode phase: -120°
- Nichols point: $(-120^\circ, -6.02 \text{ dB})$

Frequency is still present, but as a label along the curve rather than an axis.

The critical polar point -1 becomes the Nichols point $(-180^\circ, 0 \text{ dB})$.

Nichols chart: derive the closed-loop mapping

Write the loop value as $L = re^{j\phi}$, where $r = 10^{M/20}$.

$$\begin{aligned}|T|^2 &= \left| \frac{L}{1+L} \right|^2 = \frac{r^2}{|1+re^{j\phi}|^2}, \\ |1+re^{j\phi}|^2 &= (1+r\cos\phi)^2 + (r\sin\phi)^2 \\ &= 1+r^2+2r\cos\phi.\end{aligned}$$

Therefore,

$$|T| = \frac{r}{\sqrt{1+r^2+2r\cos\phi}}.$$

Every Nichols coordinate implies a closed-loop magnitude for unity feedback.

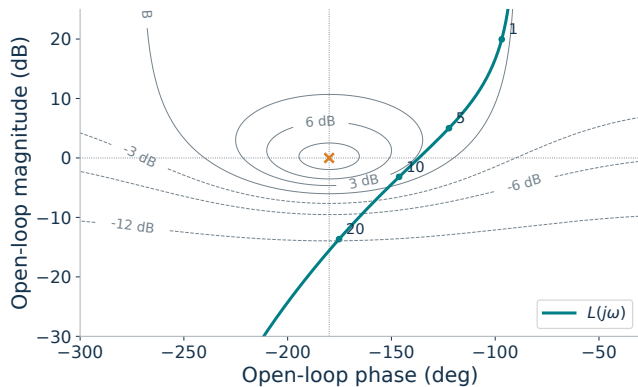
Constant- $|T|$ contours

A contour joins loop values that give the same closed-loop magnitude.

The grey labels are values of $20 \log_{10} |T|$.

The teal curve is the loop from the margin example. Numbers on it label ω in rad/s.

Near $(-180^\circ, 0)$, a small denominator produces large amplification.



Follow the loop through the contours to read the closed-loop response.

Read margins and bandwidth on Nichols axes

Use different references for different specifications.

- ① Phase margin: at the loop's 0 dB crossing, measure the horizontal distance from -180° .
- ② Gain margin: at the -180° crossing, measure the vertical distance to 0 dB.
- ③ Resonant peak: find the highest closed-loop magnitude contour reached by the loop.
- ④ Bandwidth: find the labeled frequency where the loop crosses the contour $20 \log_{10} |T(0)| - 3.01$ dB.

A Nichols axis gives open-loop data; a contour gives closed-loop data.

Without frequency labels, a Nichols curve does not by itself identify the bandwidth in rad/s.

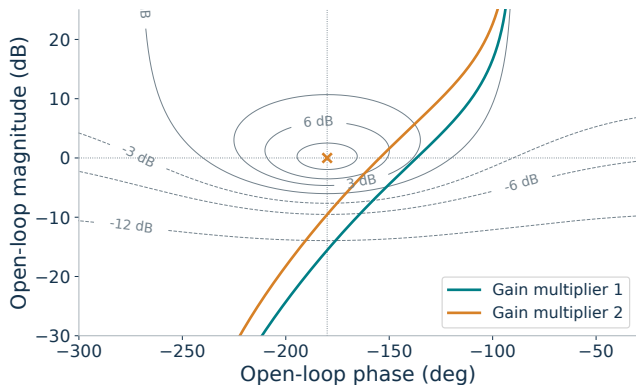
Changing gain shifts the Nichols curve vertically

If $L_{\text{new}} = kL$, then

$$M_{\text{new}} = M + 20 \log_{10} k,$$

$$\phi_{\text{new}} = \phi.$$

Doubling the gain shifts every point upward by 6.02 dB. The new curve reaches different closed-loop contours and crosses 0 dB at a different phase.



Increasing gain changes both speed and damping, even though it adds no phase.

Nichols example: calculate one loop point

Take $L = 0.5\angle(-120^\circ)$ from page 38.

Since $\cos(-120^\circ) = -1/2$,

$$|T| = \frac{0.5}{\sqrt{1 + 0.5^2 + 2(0.5)(-1/2)}} = \frac{0.5}{\sqrt{0.75}} = 0.577.$$

Thus the point lies on the $20 \log_{10}(0.577) = -4.77$ dB closed-loop contour. For phase,

$1 + L = 0.75 - 0.433j$, whose angle is -30° ; therefore

$$\angle T = \angle L - \angle(1 + L) = -120^\circ - (-30^\circ) = -90^\circ.$$

The loop's -6.02 dB magnitude is not the closed-loop magnitude.

Design begins with a measurable specification

Choose the response you need before choosing a compensator.

- ① Specify tracking error, response speed, overshoot, disturbances, and noise.
- ② Inspect the plant model: integrators, resonances, delays, and uncertain dynamics.
- ③ Choose a candidate crossover and required phase margin.
- ④ Shape $L = CP$ with a controller structure and tune its parameters.
- ⑤ Recompute the full closed-loop poles, T , S , and time response.

A phase-margin target is a design aid. The actual time response remains the final check.

The next example walks through every parameter of one lead controller.

Lead compensation near crossover

Place the zero below the pole.

$$C_{\text{lead}}(s) = K_c \frac{1 + \tau s}{1 + \alpha \tau s}, \quad 0 < \alpha < 1.$$

Its phase is $\phi = \tan^{-1}(\omega\tau) - \tan^{-1}(\alpha\omega\tau)$. Setting $d\phi/d\omega = 0$ gives $\omega\tau = 1/\sqrt{\alpha}$; substitution gives

$$\omega_m = \frac{1}{\tau\sqrt{\alpha}}, \quad \sin \phi_{\text{max}} = \frac{1 - \alpha}{1 + \alpha}, \quad |C_{\text{lead}}(j\omega_m)| = \frac{K_c}{\sqrt{\alpha}}.$$

α sets the phase advance; τ places its peak; K_c sets the magnitude.

Phase and magnitude change together; tune them together.

Lead design: find the phase shortfall

Plant: $P(s) = 10/[(1 + 0.2s)(1 + 0.5s)]$.

Design targets: $\omega_{gc} = 10$ rad/s and $PM \geq 55^\circ$.

$$\angle P(j10) = -\tan^{-1}(2) - \tan^{-1}(5) = -142.13^\circ,$$

$$|P(j10)| = \frac{10}{\sqrt{5}\sqrt{26}} = 0.8771.$$

If gain alone forced crossover to 10, the phase margin would be

$$180^\circ - 142.13^\circ = 37.87^\circ.$$

We need at least 17.13° additional phase; choose 22° for a nominal margin near 60° .

This calculation determines what the lead must supply at the chosen crossover.

Lead design: calculate all three parameters

Place the maximum 22° phase advance at 10 rad/s.

$$\alpha = \frac{1 - \sin 22^\circ}{1 + \sin 22^\circ} = 0.45496,$$

$$\tau = \frac{1}{10\sqrt{\alpha}} = 0.14826 \text{ s},$$

$$K_c = \frac{\sqrt{\alpha}}{|P(j10)|} = \frac{0.67451}{0.87706} = 0.76906.$$

Therefore,

$$C(s) = 0.76906 \frac{1 + 0.14826s}{1 + 0.06745s}.$$

Zero: 6.745 rad/s. Pole: 14.826 rad/s.

The magnitude condition is exact: $|C(j10)P(j10)| = 1$.

Lead example: verify the new loop

Check the exact loop, not a hand-drawn sketch.

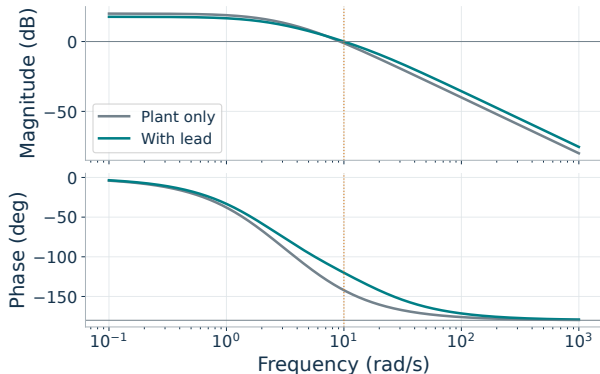
At $\omega = 10$ rad/s,

$$|L| = 1,$$

$$\angle L = -142.13^\circ + 22^\circ = -120.13^\circ,$$

$$\boxed{\text{PM} = 59.87^\circ}.$$

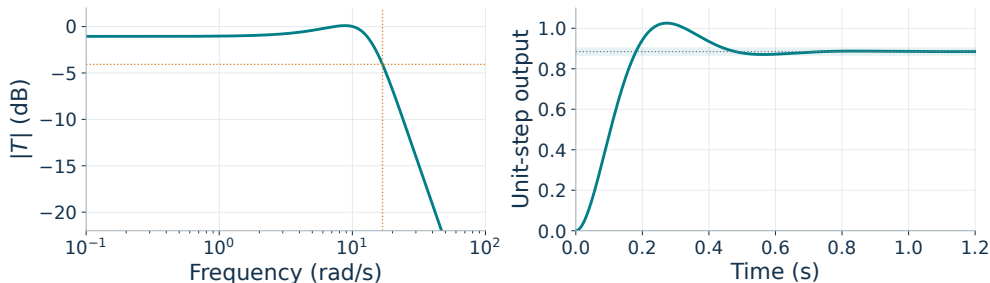
The computed closed-loop poles all have negative real parts.



Both stated loop targets are met. Now test what they imply for tracking.

Lead design: verify closed-loop tracking

The loop targets do not uniquely determine closed-loop bandwidth or overshoot.



$$T(0) = 0.8849 \quad \omega_B = 16.84 \text{ rad/s} \quad \text{Overshoot} = 15.88\% \quad T_s(2\%) = 0.443 \text{ s}$$

$$\text{For a unit step, } e_\infty = 1/[1 + L(0)] = 1/(1 + 7.6906) = 11.51\%.$$

There is no integrator. Bandwidth, overshoot, and settling are relative to the final value.

A lag compensator increases low-frequency gain

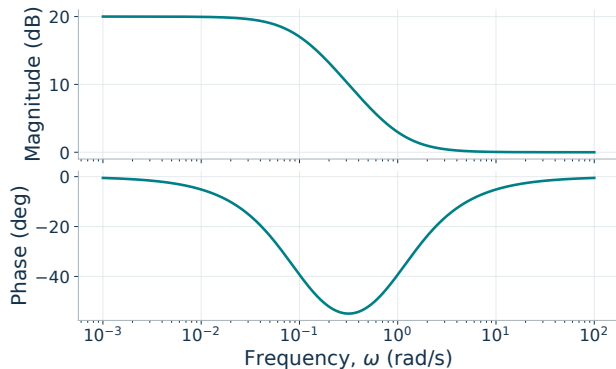
Keep the high-frequency gain near one.

$$C_{\text{lag}}(s) = \beta \frac{1 + s/\omega_z}{1 + s/\omega_p},$$

where $0 < \omega_p < \omega_z$ and $\beta = \omega_z/\omega_p > 1$.

$$C_{\text{lag}}(0) = \beta, \quad C_{\text{lag}}(\infty) = 1.$$

Example: $\beta = 10$, $\omega_p = 0.1$, $\omega_z = 1$ rad/s.



Place the pole and zero well below crossover to limit the added phase lag there.

Combine accuracy and phase requirements

Lead-lag

Use lead around crossover and lag at lower frequencies:

$$C = C_{\text{lead}} C_{\text{lag}}.$$

Increasing $L(0)$ reduces $e_{\infty} = 1/[1 + L(0)]$, but lag can move crossover. Recheck phase margin after combining them.

Accuracy, stability, actuator limits, and noise must be checked in the same final model.

Use anti-windup if actuator saturation can occur. Avoid relying on fragile pole-zero cancellation.

Integral action

For a stable unity-feedback loop, adding an uncanceled pole at the origin can eliminate step error:

$$C_I(s) = \frac{K_I}{s}, \quad S(0) = 0.$$

The integrator adds -90° phase; compensate and verify stability.

A notch targets a flexible resonance

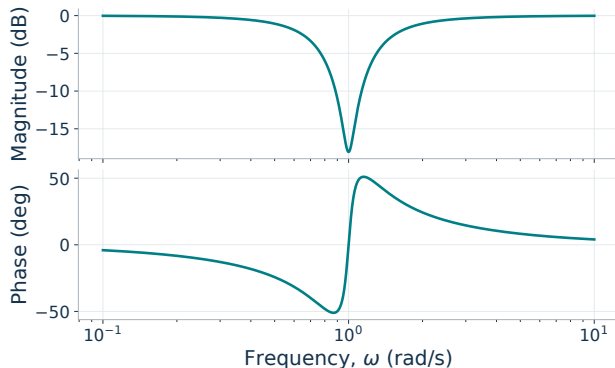
Reduce gain near a known mode.

$$C_N(s) = \frac{s^2 + 2\zeta_z\omega_r s + \omega_r^2}{s^2 + 2\zeta_p\omega_r s + \omega_r^2},$$

with $0 < \zeta_z < \zeta_p$. At its center,

$$|C_N(j\omega_r)| = \zeta_z/\zeta_p.$$

The plot uses $\omega_r = 1$ rad/s. At its center, $0.05/0.4 = 0.125$, or -18.06 dB.



A notch does not automatically improve phase margin; verify the new crossings and mode uncertainty.

Textbook case: photovoltaic voltage control

Example 8.6 specifies low-frequency rejection and high-frequency attenuation.

$$P(s) = \frac{300000}{s(s + 360)}, \quad C(s) = 250 \frac{1 + 0.04s}{1 + 100s}.$$

Write $M_L = 20 \log_{10} |L|$. The targets are $M_L \geq 20$ dB for $\omega \leq 10$ rad/s, $M_L \leq -20$ dB for $\omega \geq 1000$ rad/s, and $\text{PM} \geq 60^\circ$.

- Controller pole: 0.01 rad/s; zero: 25 rad/s.
- Therefore this controller is a lag compensator.
- Its improved phase margin comes mainly from lowering crossover, not from adding positive phase.

Judge the controller by its pole–zero locations and the resulting loop.

Photovoltaic case: check the frequency targets

Calculated with the stated textbook model.

At 10 rad/s:

$$M_L = 27.02 \text{ dB} \geq 20 \text{ dB}.$$

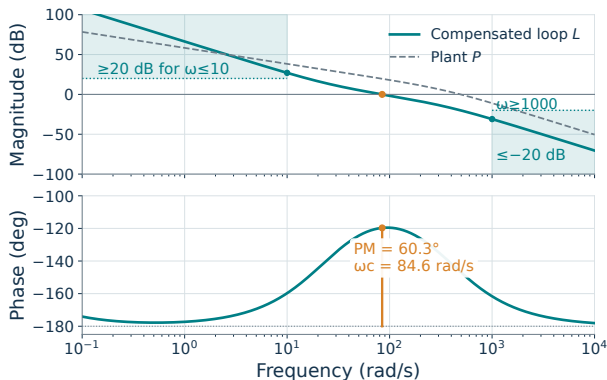
At 1000 rad/s:

$$M_L = -30.98 \text{ dB} \leq -20 \text{ dB}.$$

$$\omega_{gc} = 84.59 \text{ rad/s}$$

$$\text{PM} = 60.32^\circ.$$

All three frequency targets are satisfied.



The loop magnitude is monotone decreasing, so the endpoint checks establish the two specified frequency bands.

Textbook case: position a robot leg

Example 8.7 mixes frequency- and time-domain requirements.

$$P(s) = \frac{1}{s(s^2 + 2s + 10)}, \quad C(s) = K \frac{s^2 + as + b}{s + c}.$$

Required: bandwidth at least 1 Hz = 2π rad/s; step overshoot at most 15%; zero step tracking error.

- ① The plant's integrator can provide zero step error if the final loop is stable.
- ② The second-order overshoot relation suggests $\zeta \geq 0.52$.
- ③ The textbook uses the approximate second-order bandwidth fit

$$\omega_B \approx (1.8508 - 1.1961\zeta)\omega_n$$

$$\text{so } \omega_n = 2\pi / (1.8508 - 1.1961 \times 0.52) = 5.113 \text{ rad/s.}$$

These are initial pole targets; the full fourth-order response still needs verification.

Robot-leg design: match the coefficients

Choose the two extra poles at $-q$, with $q = 12\zeta\omega_n = 31.906$.

The target is $(s^2 + 2\zeta\omega_n s + \omega_n^2)(s + q)^2$. Match it to

$$s^4 + (2 + c)s^3 + (10 + 2c + K)s^2 + (10c + Ka)s + Kb.$$

This gives, in calculation order,

$$c = 2\zeta\omega_n + 2q - 2 = 67.13,$$

$$K = \omega_n^2 + 4\zeta\omega_n q + q^2 - 10 - 2c = 1239.21,$$

$$a = (2q\omega_n^2 + 2\zeta\omega_n q^2 - 10c)/K = 5.173,$$

$$b = \omega_n^2 q^2 / K = 21.477.$$

Rounded controller: $C(s) = 1239(s^2 + 5.17s + 21.48)/(s + 67.13)$.

Retain the full precision of ω_n during the calculation. The extra poles lie well to the left of the desired dominant pair.

Robot-leg design: verify the full response

Compare the designed system with the second-order starting target.



$27.26 > 2\pi$ rad/s, $14.42\% < 15\%$, and $T(0) = 1$: all three targets pass.

Curves use the rounded controller coefficients printed in the textbook. The ideal differentiating controller requires high-frequency roll-off for a physical implementation.

A reproducible software workflow

Software checks calculations; it does not choose the correct transfer function for you.

- 1 Enter P and C , form $L = CP$ and the negative-feedback response $T = L/(1 + L)$.
- 2 Verify the poles of T before interpreting its steady-state response.
- 3 Evaluate a logarithmic frequency grid; refine it near crossings and resonances.
- 4 Calculate margins from L and bandwidth or peaking from T .
- 5 Simulate the step response long enough to include slow tails.

Compare at least one software value with a hand calculation.

The accompanying Python and MATLAB scripts reproduce the examples; all frequencies are in rad/s unless a hertz conversion is stated.

Engraving-machine design: choose the gain

Example 8.8 starts with proportional gain $K = 2$.

$$P(s) = \frac{1}{s(s+1)(s+2)}, \quad T(s) = \frac{K}{s^3 + 3s^2 + 2s + K}.$$

The Routh first column is 1, 3, $(6 - K)/3$, K , so stability requires $0 < K < 6$.

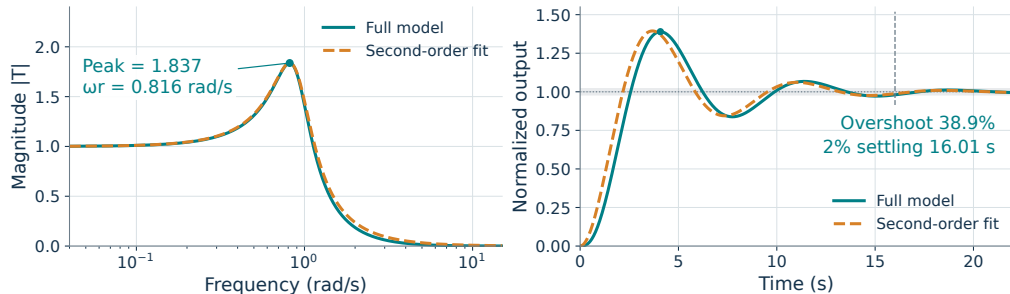
- ① Compute the closed-loop resonant peak M_r and its frequency ω_r .
- ② If a second-order approximation is reasonable, infer

$$\zeta = \sqrt{\frac{1 - \sqrt{1 - 1/M_r^2}}{2}}, \quad \omega_n = \frac{\omega_r}{\sqrt{1 - 2\zeta^2}}.$$

- ③ Predict overshoot and settling, then compare with the actual step response.

Engraving machine: estimate versus simulation

$M_r = 1.837$ and $\omega_r = 0.8165$ give $\zeta = 0.28384$ and $\omega_n = 0.89147$.



Predicted overshoot: 39.46%. Actual: 38.94%. Actual settling: 16.01 s.

The orange curve uses the inferred second-order model. Settling uses a 2% band.

Disk-drive model: include the flexible mount

A neglected resonance can invalidate a high-bandwidth design.

Let G_1 model the motor coil, G_2 the arm, and G_3 the flexure and head:

$$C(s) = 400(s + 1), \quad G_1(s) = \frac{5}{1 + 0.001s}, \quad G_2(s) = \frac{0.05}{s(1 + 0.05s)},$$

$$G_3(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \quad \zeta = 0.3, \quad \omega_n = 18850 \text{ rad/s}.$$

Form the loop $L = CG_1G_2G_3$ and the tracking response $T = L/(1 + L)$.

The flexure is far above crossover, but its effect still needs checking.

G_3 is normalized to unit DC gain; static compliance is absorbed into the surrounding plant gain.

Disk drive: crossover and bandwidth

Exact calculations for $K = 400$.

Loop crossover:

$$\omega_{gc} = 1252 \text{ rad/s.}$$

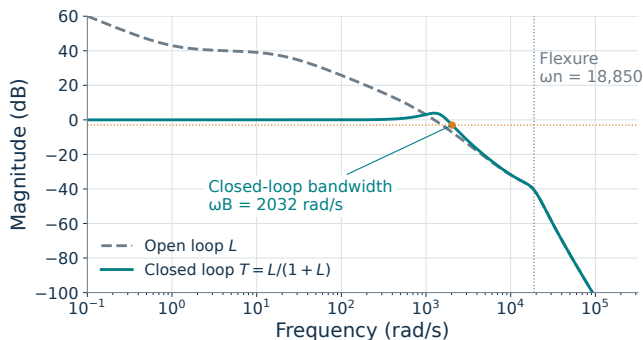
Phase margin:

$$\text{PM} = 37.18^\circ.$$

Tracking bandwidth:

$$\omega_B = 2032 \text{ rad/s.}$$

The isolated flexure peaks at $0.906\omega_n$ with 4.85 dB amplification.



Bandwidth and crossover are different numbers, even in this textbook example.

Disk drive: verify the settling time

Verify the full model in time.

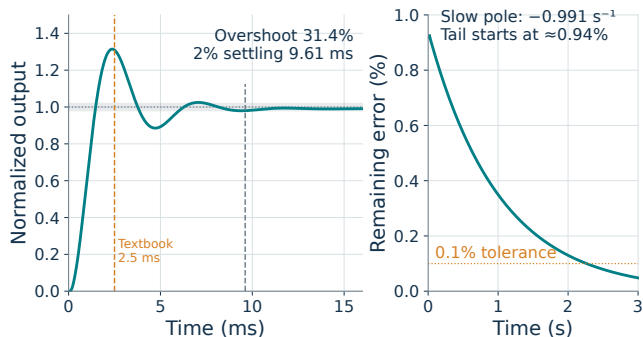
Calculated overshoot:
31.44%.

Calculated 2% settling:
9.61 ms.

The textbook's 2.5 ms estimate assumes much stronger damping than the full model has.

At 2.5 ms, the output is still about 1.31.

A slow tail below 1% does not govern 2% settling, but matters for tighter accuracy.



Measure a plant with feedback operating

Use an independent reference injection when opening the loop is unsafe.

Record r , plant input u , and output y . For disturbances uncorrelated with r ,

$$G_{yr} = \frac{S_{yr}}{S_{rr}} = T, \quad G_{ur} = \frac{S_{ur}}{S_{rr}} = CS.$$

Because $T = PCS$,

$$P = \frac{G_{yr}}{G_{ur}} = \frac{S_{yr}}{S_{ur}}.$$

If $G_{yr} = 0.6 - 0.2j$ and $G_{ur} = 0.4$, then $P = 1.5 - 0.5j$.

Use the independent injected signal as the reference; direct Y/U can be biased in feedback.

Requirements: synchronized signals, sufficient excitation, and nonzero G_{ur} .

Implementation checks

The model must include the dynamics that can limit the real controller.

- Sampling and computation: include delay and the discretized controller.
- Actuator limits: check saturation, rate limits, and integrator windup.
- Sensor noise: inspect high-frequency T and control effort.
- Flexible modes: perturb resonant frequency and damping, not only gain.
- Model uncertainty: check plausible parameter ranges and all relevant crossings.

A nominal Bode plot is one test, not the entire validation.

A high-order loop with multiple crossings or unstable open-loop poles needs a complete stability analysis, such as Nyquist, in addition to scalar margins.

Identify a model from measured Bode slopes

Suppose the measured plant has a 20 dB DC gain and breaks at 2 and 20 rad/s.

The slope changes $0 \rightarrow -20 \rightarrow -40$ dB/decade, with no resonant peak.

- 1 DC magnitude gives $K = 10^{20/20} = 10$.
- 2 Each -20 dB/decade change suggests one real pole.
- 3 A candidate stable minimum-phase model is

$$\hat{P}(s) = \frac{10}{(1 + s/2)(1 + s/20)}.$$

- 4 At 2 rad/s it predicts 16.95 dB and -50.71° ; compare both with the data.

Slopes suggest a model. Phase and extra test points can reject it.

This identification assumes no hidden delay or right-half-plane zeros; magnitude alone cannot establish those assumptions.

Calculation check: margins and bandwidth

Try these three calculations.

- 1 At a gain crossover, phase is -130° . Find PM.
- 2 At a phase crossover, magnitude is -8 dB. Find GM as a factor.
- 3 A first-order $T = 1/(1 + 0.2s)$ tracks a step. Find ω_B and 10–90% rise time.

Worked answers

1. $PM = 180 - 130 = 50^\circ$.
2. $GM = 10^{8/20} = 2.512$.
3. $\omega_B = 1/0.2 = 5$ rad/s.
 $t_{10-90} = 0.2 \ln 9 = 0.439$ s.

The last relation is exact for this first-order model, not a universal conversion.

The complete reasoning chain

Each representation answers a different part of the same design problem.

- ① A stable LTI model maps a sinusoid to a new amplitude and phase.
- ② Factoring the model makes those changes easy to build in Bode form.
- ③ Measurement tests whether the model matches the physical plant.
- ④ Feedback maps the loop L into tracking T and sensitivity S .
- ⑤ Margins and Nichols contours guide controller choices.
- ⑥ The complete model and time response verify whether the specifications are met.

Keep the model, the plotted quantity, and the specification aligned.

Further practice: design, predict, then verify

Use the supplied scripts as checks after your own calculations.

- 1 Sketch $G(s) = 5(1 + s/2)/[s(1 + s/10)]$. Verify magnitude and phase at $\omega = 2$ and 10 rad/s.
- 2 For the lead-design plant, redesign for $\omega_{gc} = 8$ rad/s and $\text{PM} \geq 60^\circ$. Compare the actual bandwidth and step error.
- 3 Add a lag stage to reduce the lead example's step error below 2%. Recheck all loop and tracking specifications.
- 4 In the disk-drive example, vary flexure damping from 0.2 to 0.4. Explain which frequency and time metrics change most.

State assumptions and units, show one hand check, and report any specification that fails.

References and notation

Primary course reference

Richard C. Dorf and Robert H. Bishop, *Modern Control Systems*, 14th edition, Chapter 8, “Frequency Response Methods,” pp. 545–593; exercises and design problems follow.

How the material is used

Core frequency-response methods and the photovoltaic, robot-leg, engraving-machine, and disk-drive models follow the chapter. Spectral FRF estimation, sensitivity, and detailed compensation procedures expand the supplied lecture material.

Notation used in this deck

$M_r = \max_{\omega} |T(j\omega)|$ is a frequency-response peak; %OS is step overshoot. ω_{gc} is loop gain crossover; ω_B is closed-loop bandwidth. All angular frequencies are rad/s unless stated otherwise.

All plotted numerical curves are calculated from the stated models.