

Homework Solutions

Modern Control Systems, 14th Global Edition

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28 selected exercises and problems from Chapters 1–7

Unless a problem states otherwise, transfer functions use zero initial conditions and negative feedback. Asymptotic stability means all characteristic roots lie strictly in the left half-plane. Normalized sensitivity is

$$S_a^T(s) = \frac{a}{T(s)} \frac{\partial T(s)}{\partial a}.$$

For a dominant underdamped pair $s = -\sigma \pm j\omega_d$,

$$\omega_n = \sqrt{\sigma^2 + \omega_d^2}, \quad \zeta = \frac{\sigma}{\omega_n}, \quad \text{PO} \approx 100e^{-\pi\sigma/\omega_d}\%, \quad T_s \approx \frac{4}{\sigma}.$$

The last relation is a 2% settling-time approximation. An *actual* settling time is the last crossing into the band $|y(t) - y_\infty| \leq 0.02|y_\infty|$ after which the response remains there. For positive final values this is $0.98y_\infty \leq y(t) \leq 1.02y_\infty$. Approximate and actual values are distinguished throughout.

Included questions

Chapter	Exercise/problem identifiers
1	E1.2, E1.7, E1.12, P1.5, P1.6, P1.9
2	E2.2, E2.8, E2.16, P2.1, P2.3, P2.8
3	E3.2, P3.12, P3.17
4	P4.7, P4.8, P4.11
5	E5.10, E5.15, P5.2, P5.13
6	E6.13, E6.15, P6.5, P6.13
7	E7.13, E7.26

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1 Chapter 1: Introduction to control systems

1.1 E1.2: Energy-conversion devices

Source: textbook printed page 67.

Problem in brief. Give examples of devices converting mechanical energy to fluid energy, mechanical energy to electrical energy, electrical energy to mechanical energy, kinetic energy to electrical energy, and electrical energy to heat.

Detailed solution. A device should be identified together with the physical input and output, not simply named.

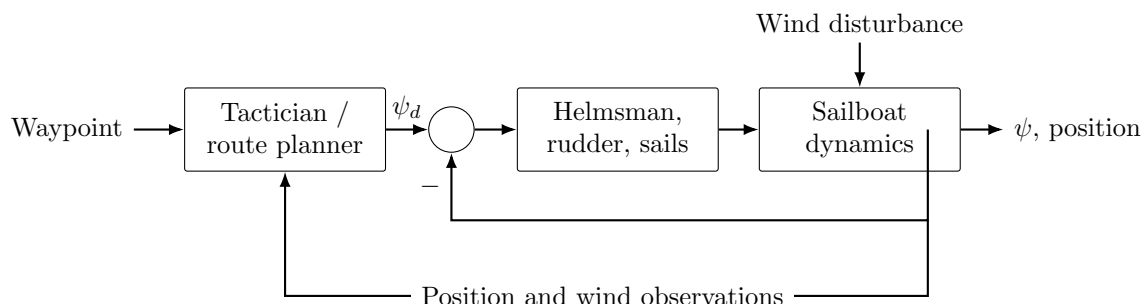
- (a) **Mechanical to fluidic: a pump or compressor.** Shaft torque at angular speed ω supplies mechanical power $P_{\text{in}} = \tau\omega$. A hydraulic pump produces flow Q against a pressure difference Δp , with fluid power $P_{\text{out}} = \Delta p Q$. A compressor produces pressurized gas.
- (b) **Mechanical to electrical: a generator.** Rotating a conductor or magnetic field induces an emf; mechanical shaft power becomes electrical terminal power vi . For an ideal permanent-magnet machine, $e = K_e\omega$ and electromagnetic torque is proportional to current.
- (c) **Electrical to mechanical: an electric motor.** Applied voltage and current create electromagnetic torque and shaft rotation. A solenoid is a corresponding linear-motion example: current produces a force and displacement.
- (d) **Kinetic to electrical: a turbine-generator or regenerative-braking generator.** A wind turbine extracts moving-air kinetic energy, or a braking motor extracts vehicle kinetic energy, and a generator converts the resulting shaft power to electricity. Kinetic energy is a form of mechanical energy, so overlap with part (b) is physically appropriate.
- (e) **Electrical to heat: a resistance heater.** Joule heating gives $P = i^2R = v^2/R$. A heating element converts electrical input into thermal energy delivered to the controlled body.

1.2 E1.7: Sailboat tacking under shifting wind

Source: textbook printed page 67.

Problem in brief. Explain how a sailboat adjusts its tacking strategy when the wind changes, and represent the process by a feedback block diagram.

Detailed solution. A tactician selects a feasible heading on either side of the wind, outside the no-sail zone. The boat follows successive port and starboard tacks to progress toward the waypoint. Wind shifts change the favored tack, so the tactician updates the chosen heading using wind direction, boat position, and progress toward the waypoint. The helmsman then compares the commanded and measured heading and moves the rudder; sail trim is adjusted to maintain propulsion.



The inner loop regulates heading ψ . The outer loop selects a tack and heading to reduce route error. A tack changes the side on which the wind is received by turning the bow through the wind; it is not

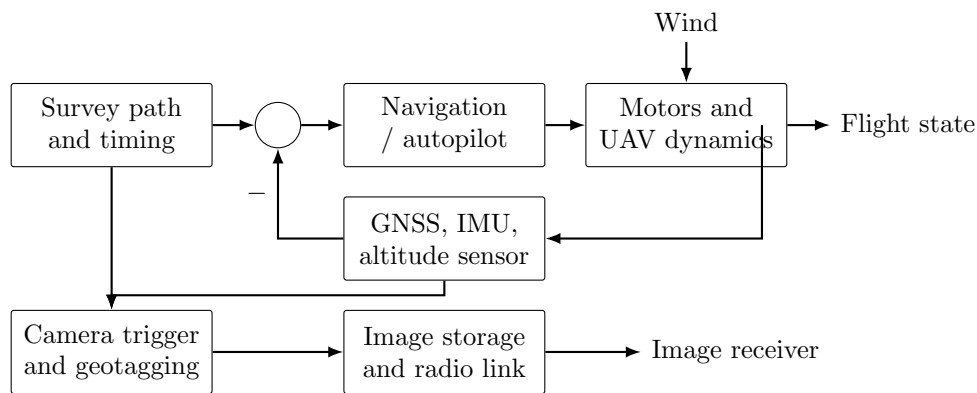
a command to sail straight into the wind. Hysteresis or a minimum tack duration avoids repeated switching in noisy wind measurements.

1.3 E1.12: Autonomous UAV crop photography

Source: textbook printed page 67.

Problem in brief. Propose an autonomous UAV architecture that accurately follows a prescribed survey trajectory, photographs the entire field, and transmits the photographs without human ground control.

Detailed solution. The mission computer stores the field boundary and a survey path, usually parallel strips with sufficient image overlap. A trajectory generator supplies desired position, velocity, altitude, and heading as functions of time. Navigation combines GNSS with inertial and altitude measurements. Position errors generate attitude/thrust commands, and a faster attitude loop stabilizes the aircraft.



The camera is triggered by position or distance traveled, not merely by a free-running timer. Images are geotagged, stored, and transmitted; onboard coverage tracking can schedule a repeat pass if an area was missed. The ground station receives data but does not close the flight-control loop. Battery and obstacle checks can initiate a predefined return or safe landing autonomously.

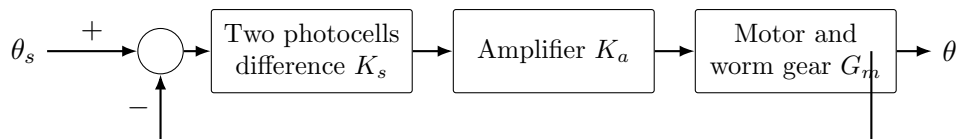
1.4 P1.5: Two-photocell sun tracker

Source: textbook printed page 69.

Problem in brief. Complete the feedback arrangement for a motor and reduction gear carrying two photocells, so that the assembly follows the light source.

Detailed solution. Let θ_s be the light-source direction and θ the bracket angle. The two shielded photocells produce equal signals only when aligned. Wire their difference as $e_L = V_L - V_R$, choosing polarity so that a positive angular error $\theta_s - \theta$ produces motor rotation that reduces that error. Near alignment,

$$e_L \approx K_s(\theta_s - \theta), \quad K_s > 0.$$



Here the summing junction is a conceptual representation of the optical angular comparison; no separate electronic measurement of θ_s is needed. The physical bracket rotates both sensors, closing

the loop. For the linearized model,

$$\frac{\Theta(s)}{\Theta_s(s)} = \frac{K_s K_a G_m(s)}{1 + K_s K_a G_m(s)}.$$

When the cell signals become equal, the motor command tends to zero. Reversing either the cell subtraction or motor wiring incorrectly can create positive feedback. Gain should be limited to avoid hunting due to gear inertia and motor lag.

1.5 P1.6: The wage–price feedback spiral

Source: textbook printed page 69.

Problem in brief. Explain the positive wage–price loop, add stabilizing policy feedback, and discuss altered or delayed cost-of-living information and national wage/price guidelines.

Detailed solution. Let w_0 denote the initial wage command, w actual wages, and p prices. The automatic cost-of-living adjustment increases wages when prices rise:

$$w = w_0 + K_1 p, \quad p = P(s)w.$$

Consequently, $p/w_0 = P/(1 - K_1 P)$: the positive feedback subtracts from the characteristic denominator. For the illustrative model $P(s) = b/(\tau s + 1)$,

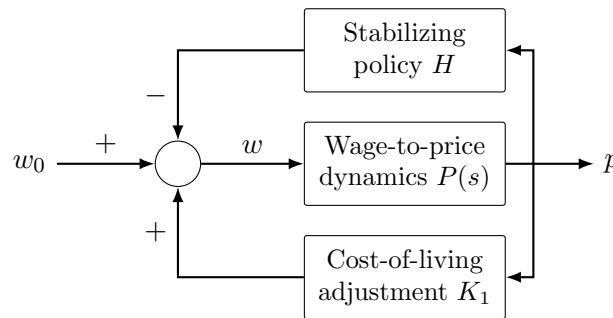
$$\tau s + 1 - bK_1 = 0, \quad \text{stability requires } bK_1 < 1.$$

A stabilizing policy loop

An additional policy signal $u_p = -Hp$ can represent a reduction in effective wage/price pressure when inflation rises. Taxes, monetary restraint, or negotiated limits enter through a policy actuator with the appropriate sign. The model becomes

$$w = w_0 + K_1 p - Hp, \quad \frac{p}{w_0} = \frac{P}{1 + P(H - K_1)}.$$

For constant H and the simple plant above, stability requires $1 + b(H - K_1) > 0$.



Information changes and guidelines

If a reported price index is scaled to $p_m = \alpha p$, with $0 < \alpha < 1$, the effective positive-loop gain becomes αK_1 . In the illustrative first-order model this can stabilize the mathematical loop if $b\alpha K_1 < 1$. Merely adding a constant reporting bias changes the equilibrium but not the small-signal loop gain, so it does not by itself fix instability.

A reporting delay gives characteristic equation $\tau s + 1 - bK_1 e^{-sT} = 0$. If $bK_1 > 1$, its left side is negative at real $s = 0$ and positive for sufficiently large positive real s , so a positive real root exists for every delay $T \geq 0$. Delaying the data therefore cannot cure this excessive positive DC gain; it can also impair other feedback loops. Wage/price guidelines instead reduce the effective wage-indexing gain, constrain price-setting behavior, or add explicit negative feedback. These are qualitative control models; policy effectiveness depends on the actual economic dynamics.

1.6 P1.9: Physiological feedback models

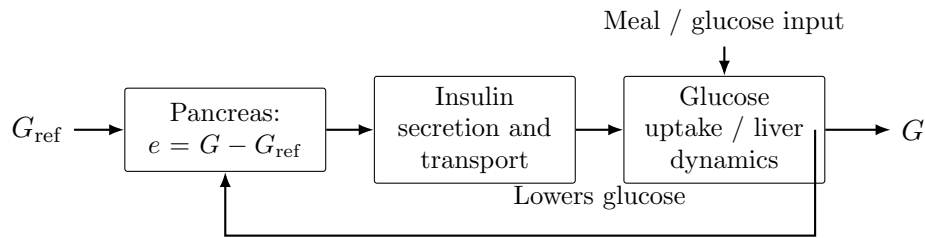
Source: textbook printed page 69–70.

Problem in brief. Review the heart-rate block model, identify useful refinements, and construct a model for one of the six listed physiological control systems.

Detailed solution. The supplied model contains a lung-stretch input to the medulla, neural control of the heart, conversion of heart rate into vascular pressure, and pressure-receptor feedback. It is a reasonable functional starting point. A more explicit model can add a desired operating level and pressure comparator, separate sympathetic and parasympathetic pathways, include heart contractility and vascular resistance, and include neural/sensor delays. The lung-stretch pathway is retained as a respiratory influence rather than mistaken for the pressure feedback signal. The variables represent vectors, so replacing each physiological subsystem by an unexplained scalar constant may miss important coupling.

One admissible choice: pancreas and blood-glucose regulation

The regulated variable is blood glucose. Pancreatic sensing compares it with the normal operating level; insulin secretion is the control action. Insulin promotes glucose uptake and suppresses hepatic production, reducing elevated blood glucose. A meal is an external disturbance. A schematic using $e = G - G_{\text{ref}}$ makes the sign explicit:



For a simple local model, let $g = G - G_0$ and $i = I - I_0$ be deviations from equilibrium. A consistent pair of equations is

$$\dot{g} = -ag - bi + d, \quad \dot{i} = cg - fi, \quad a, b, c, f > 0.$$

Here d is a glucose-input disturbance, cg is increased secretion for increased glucose, and $-bi$ is insulin's glucose-lowering effect. Thus

$$A = \begin{bmatrix} -a & -b \\ c & -f \end{bmatrix}, \quad q(s) = s^2 + (a + f)s + af + bc.$$

All coefficients are positive, giving a stable local negative-feedback model. Glucagon and other hormonal pathways can be added to describe low-glucose regulation more faithfully. The equations are a proposed qualitative model, not numerical patient data specified by the exercise.

2 Chapter 2: Mathematical models of systems

2.1 E2.2: Thermistor linearization

Source: textbook printed page 158.

Problem in brief. Linearize $R(T) = R_0 e^{-0.3T}$ with $R_0 = 5000 \Omega$ near $T_0 = 20^\circ\text{C}$. The printed exponential uses a lowercase t , but the statement defines the independent variable as temperature T .

Detailed solution. The equilibrium resistance is

$$R_* = R(20) = 5000e^{-6} = 12.3938 \Omega.$$

The derivative at the operating temperature is

$$\left. \frac{dR}{dT} \right|_{20} = -0.3(5000)e^{-6} = -3.71813 \Omega/^\circ\text{C}.$$

For $\Delta T = T - 20$ and $\Delta R = R - R_*$, the first-order Taylor model is

$$\Delta R \approx -3.71813\Delta T, \quad R(T) \approx 12.3938 - 3.71813(T - 20) \Omega.$$

It is valid for $|0.3\Delta T| \ll 1$. This reproduces the book's rounded incremental relation $\Delta R \approx -3.7\Delta T$. The constant equilibrium resistance must be included when expressing absolute R rather than ΔR .

2.2 E2.8: The multiloop ship-steering system

Source: textbook printed page 159.

Problem in brief. Find $Y(s)/R(s)$ for the ship-steering diagram with forward blocks $K, G_1, G_2, 1/s$ and the three feedback blocks H_1, H_2, H_3 .

Detailed solution. Let A be the rudder angle, and let $V = G_2 A$ be the signal immediately before the heading integrator, so $Y = V/s$. The outer command is $K(R - Y)$. From the original diagram, H_1 and H_2 both receive V , and H_3 receives A ; all three local feedback contributions are subtracted. Therefore

$$A = G_1 [K(R - Y) - H_1 V - H_2 V - H_3 A].$$

Substitute $V = G_2 A$ and $Y = G_2 A/s$:

$$A \left[1 + G_1 H_3 + G_1 G_2 (H_1 + H_2) + \frac{K G_1 G_2}{s} \right] = K G_1 R.$$

Multiplication by G_2/s yields

$$\frac{Y}{R} = \frac{K G_1 G_2 / s}{1 + G_1 H_3 + G_1 G_2 (H_1 + H_2) + K G_1 G_2 / s}.$$

Equivalently,

$$\frac{Y}{R} = \frac{K G_1 G_2}{s[1 + G_1 H_3 + G_1 G_2 (H_1 + H_2)] + K G_1 G_2}.$$

Every block here is evaluated at s . Setting all H_i to zero leaves the ordinary unity-feedback loop $K G_1 G_2 / s$, which checks the expression.

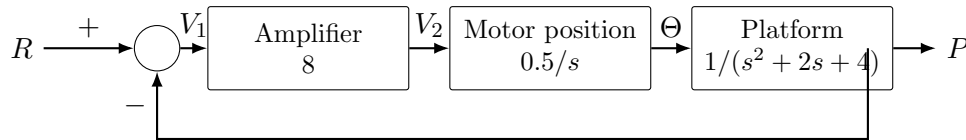
2.3 E2.16: Spacecraft platform position control

Source: textbook printed page 161.

Problem in brief. The platform satisfies $\ddot{p} + 2\dot{p} + 4p = \theta$, $v_1 = r - p$, $\dot{\theta} = 0.5v_2$, and $v_2 = 8v_1$. Draw the component interconnection and determine $P(s)/R(s)$.

Detailed solution. The amplifier, motor-position, and platform transfer functions are

$$\frac{V_2}{V_1} = 8, \quad \frac{\Theta}{V_2} = \frac{0.5}{s}, \quad \frac{P}{\Theta} = \frac{1}{s^2 + 2s + 4}.$$



The forward path is $F(s) = 4/[s(s^2 + 2s + 4)]$. Hence

$$\frac{P(s)}{R(s)} = \frac{F}{1 + F} = \frac{4}{s^3 + 2s^2 + 4s + 4}.$$

The same result follows from $\dot{\theta} = 4(r - p)$ and $\theta = \ddot{p} + 2\dot{p} + 4p$: differentiating the latter gives $p^{(3)} + 2\ddot{p} + 4\dot{p} + 4p = 4r$.

2.4 P2.1: Electrical-network mesh equations

Source: textbook printed page 164.

Problem in brief. Form simultaneous integrodifferential equations for the two-mesh circuit. The left mesh contains the source $v(t)$, R_1 , C_1 , and shared capacitor C_2 ; the right mesh contains R_2 , L_1 , R_3 , and C_2 .

Detailed solution. Use the clockwise mesh currents i_1 and i_2 shown in the book. The downward current through shared C_2 is $i_1 - i_2$. Define v_{C1} as the left-to-right voltage across C_1 , and v_{C2} as the top-to-bottom voltage across C_2 . Then

$$v_{C1}(t) = v_{C1}(0) + \frac{1}{C_1} \int_0^t i_1(\tau) d\tau,$$

$$v_{C2}(t) = v_{C2}(0) + \frac{1}{C_2} \int_0^t [i_1(\tau) - i_2(\tau)] d\tau.$$

Kirchhoff's voltage law in the left mesh gives

$$R_1 i_1 + \frac{1}{C_1} \int_0^t i_1 d\tau + \frac{1}{C_2} \int_0^t (i_1 - i_2) d\tau = v(t) - v_{C1}(0) - v_{C2}(0).$$

The right mesh traverses C_2 upward, giving

$$L_1 \dot{i}_2 + (R_2 + R_3) i_2 + \frac{1}{C_2} \int_0^t (i_2 - i_1) d\tau = v_{C2}(0).$$

For uncharged capacitors, the constant voltage terms vanish. An equivalent set of circuit equations is

$$\begin{aligned} R_1 i_1 + v_{C1} + v_{C2} &= v, & L_1 \dot{i}_2 + (R_2 + R_3) i_2 - v_{C2} &= 0, \\ \dot{v}_{C1} &= i_1/C_1, & \dot{v}_{C2} &= (i_1 - i_2)/C_2. \end{aligned}$$

These signs are fixed by the current directions; assigning opposite mesh directions is also valid if used consistently.

2.5 P2.3: Two equal masses and springs

Source: textbook printed page 165.

Problem in brief. Obtain the equations of motion for two equal masses M with a wall spring k on mass 1, a connecting spring k , and a damper b from mass 2 to the right wall. Force $F(t)$ acts on mass 1.

Detailed solution. Both displacements x_1, x_2 are positive to the right. The wall spring exerts $-kx_1$ on mass 1. The connecting spring stretches by $x_2 - x_1$, exerting $k(x_2 - x_1)$ on mass 1 and the opposite force on mass 2. The damper exerts $-b\dot{x}_2$ on mass 2. Thus

$$M\ddot{x}_1 = F - kx_1 + k(x_2 - x_1), \quad M\ddot{x}_2 = -k(x_2 - x_1) - b\dot{x}_2.$$

The required equations are

$$\boxed{M\ddot{x}_1 + 2kx_1 - kx_2 = F, \quad M\ddot{x}_2 + b\dot{x}_2 + kx_2 - kx_1 = 0.}$$

In matrix form,

$$M \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + k \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}.$$

The off-diagonal spring terms agree in sign and magnitude, as expected from equal-and-opposite coupling forces.

2.6 P2.8: Bridged-T filter

Source: textbook printed page 165.

Problem in brief. Derive the unloaded bridged-T voltage transfer function and locate its poles and zeros for $R_1 = 0.5$, $R_2 = 1$, and $C = 0.5$.

Detailed solution. Let V_m be the junction voltage between the two equal series capacitors. Resistor R_1 joins this junction to ground, and R_2 bridges the input and output. With zero initial capacitor voltages and no output loading, the two node equations are

$$sC(V_m - V_{\text{in}}) + sC(V_m - V_o) + \frac{V_m}{R_1} = 0,$$

$$sC(V_o - V_m) + \frac{V_o - V_{\text{in}}}{R_2} = 0.$$

The first equation gives $V_m = sCR_1(V_{\text{in}} + V_o)/(1 + 2sCR_1)$. The second becomes $(1 + sCR_2)V_o - sCR_2V_m = V_{\text{in}}$. Substitution and multiplication by $1 + 2sCR_1$ yield

$$[1 + (2R_1 + R_2)Cs + R_1R_2C^2s^2]V_o = [1 + 2R_1Cs + R_1R_2C^2s^2]V_{\text{in}}.$$

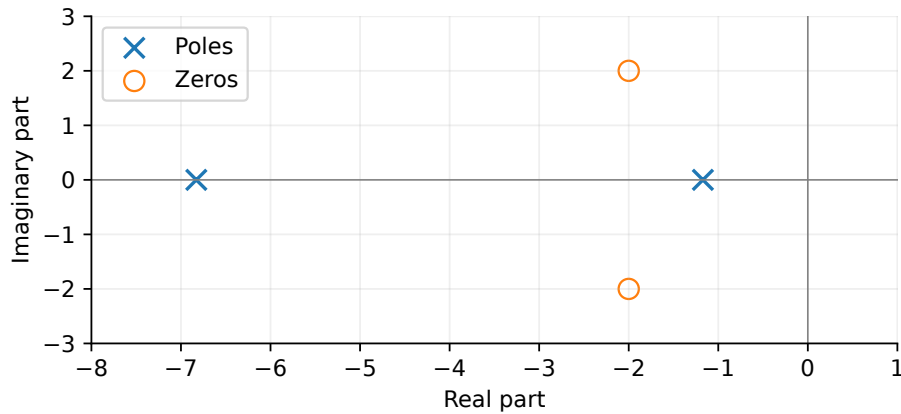
Therefore

$$\frac{V_o}{V_{\text{in}}} = \frac{1 + 2R_1Cs + R_1R_2C^2s^2}{1 + (2R_1 + R_2)Cs + R_1R_2C^2s^2}.$$

For the stated values, $R_1R_2C^2 = 1/8$, and

$$\frac{V_o}{V_{\text{in}}} = \frac{1 + 0.5s + 0.125s^2}{1 + s + 0.125s^2} = \frac{s^2 + 4s + 8}{s^2 + 8s + 8}.$$

$$\text{Zeros: } -2 \pm 2j; \quad \text{poles: } -4 \pm 2\sqrt{2} = -1.17, -6.83.$$



Crosses mark poles; circles mark zeros.

3 Chapter 3: State-variable models

3.1 E3.2: Series RLC state equations

Source: textbook printed page 239.

Problem in brief. Convert $v = Ri + L\dot{i} + v_C$ into state form, and give the numerical matrices for $R = L = C = 1$.

Detailed solution. Choose the energy variables $x_1 = v_C$ and $x_2 = i$, with input $u = v$. Since $i = C\dot{v}_C$,

$$\dot{x}_1 = x_2/C, \quad \dot{x}_2 = -x_1/L - (R/L)x_2 + u/L.$$

Hence

$$\dot{x} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} u.$$

For unit component values,

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

An output is not prescribed in the exercise. If $y = v_C$, then $C_y = [1 \ 0]$ and $D = 0$; if $y = i$, then $C_y = [0 \ 1]$. The corresponding capacitor-voltage transfer function is $1/(LCs^2 + RCs + 1)$, which checks the state equations.

3.2 P3.12: A realization and its transition matrix

Source: textbook printed page 244.

Problem in brief. Find a state model and its state transition matrix for

$$T(s) = \frac{8(s+5)}{s^3 + 12s^2 + 44s + 48}.$$

Detailed solution. Factor the denominator and expand into simple fractions:

$$s^3 + 12s^2 + 44s + 48 = (s+2)(s+4)(s+6),$$

$$T(s) = \frac{3}{s+2} - \frac{2}{s+4} - \frac{1}{s+6}.$$

For example, the residue at -2 is $8(-2+5)/[(-2+4)(-2+6)] = 3$; the other two residues follow similarly. Choose one state per first-order term:

$$\dot{x}_1 = -2x_1 + r, \quad \dot{x}_2 = -4x_2 + r, \quad \dot{x}_3 = -6x_3 + r, \quad y = 3x_1 - 2x_2 - x_3.$$

Thus a valid modal realization is

$$A = \text{diag}(-2, -4, -6), \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad C = [3 \ -2 \ -1], \quad D = 0.$$

The transition matrix is particularly simple:

$$\Phi(t) = e^{At} = \begin{bmatrix} e^{-2t} & 0 & 0 \\ 0 & e^{-4t} & 0 \\ 0 & 0 & e^{-6t} \end{bmatrix}.$$

It satisfies $\Phi(0) = I$ and $\dot{\Phi} = A\Phi$. The complete state response is

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t-\tau)B r(\tau) d\tau.$$

A controllable companion realization is also acceptable:

$$A_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -48 & -44 & -12 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C_c = [40 \quad 8 \quad 0].$$

For that realization, define

$$V = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -4 & -6 \\ 4 & 16 & 36 \end{bmatrix}; \quad \Phi_c(t) = V \operatorname{diag}(e^{-2t}, e^{-4t}, e^{-6t}) V^{-1}.$$

The two transition matrices describe different state coordinates; one must use the matrix belonging to the chosen realization.

3.3 P3.17: Transfer function from a three-state model

Source: textbook printed page 245.

Problem in brief. Given

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 4 & 3 & 0 \\ -2 & 1 & 10 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}, \quad C = [1 \quad 0 \quad 0],$$

determine $G(s) = Y(s)/U(s)$.

Detailed solution. The transformed equations are

$$(s-1)X_1 + X_2 + X_3 = 0, \quad -4X_1 + (s-3)X_2 = 0,$$

$$2X_1 - X_2 + (s-10)X_3 = 4U.$$

The second gives $X_2 = 4X_1/(s-3)$. The first then gives

$$X_3 = -\frac{(s-1)(s-3)+4}{s-3}X_1 = -\frac{s^2-4s+7}{s-3}X_1.$$

Substitute into the third and multiply by $s-3$:

$$[2(s-3) - 4 - (s-10)(s^2-4s+7)]X_1 = 4(s-3)U.$$

The bracket equals $-s^3 + 14s^2 - 45s + 60$. Since $Y = X_1$,

$$G(s) = \frac{-4(s-3)}{s^3 - 14s^2 + 45s - 60}.$$

Direct calculation gives $\det(sI - A) = s^3 - 14s^2 + 45s - 60$ and $C(sI - A)^{-1}B$ gives the same numerator. The temporary elimination denominator $s-3$ does not imply a pole at $s=3$; the final transfer function has a zero there.

4 Chapter 4: Feedback control system characteristics

4.1 P4.7: Robot joint, load torque, and nonunity sensing

Source: textbook printed page 306.

Problem in brief. The robot loop has controller k_1 , plant $P(s) = k_2/[s(\tau s + 1)]$, and sensor $H(s) = k_3 + k_4s$. Load torque $T_d = D/s$ is *subtracted* at the plant input. Find the disturbance effect, sensitivity to k_2 , and the unit-step steady-state error with no disturbance.

Detailed solution. Define the comparator error $E_a = R - HY$. The interconnection gives

$$Y = P(k_1 E_a - T_d) = P[k_1(R - HY) - T_d].$$

Consequently,

$$q(s) = \tau s^2 + (1 + k_1 k_2 k_4)s + k_1 k_2 k_3,$$

$$\boxed{\frac{Y}{R} = \frac{k_1 k_2}{q(s)}, \quad \frac{Y}{T_d} = -\frac{k_2}{q(s)}}.$$

(a) Load disturbance

For $R = 0$, $Y(s) = -k_2 D/[sq(s)]$. If q is Hurwitz, the final value is

$$\boxed{y_\infty = -\frac{D}{k_1 k_3}}.$$

Thus a positive opposing torque deflects the joint in the negative direction. A larger controller gain reduces the static deflection. The response above includes its full transient transfer function, not merely its final value.

(b) Sensitivity to plant gain

With $T = Y/R = k_1 P/(1 + k_1 P H)$ and P proportional to k_2 ,

$$\boxed{S_{k_2}^T = \frac{1}{1 + k_1 P H} = \frac{s(\tau s + 1)}{q(s)}}.$$

For example, differentiating $T = k_1 k_2/q$ directly gives the same result because $q - k_2(\partial q/\partial k_2) = s(\tau s + 1)$.

(c) The two different error definitions

For $R = 1/s$ and $T_d = 0$, $T(0) = 1/k_3$, so

$$\boxed{y_\infty = \frac{1}{k_3}, \quad e_{\text{tracking},\infty} = \lim_{t \rightarrow \infty} (r - y) = 1 - \frac{1}{k_3}}.$$

The comparator error is instead $E_a = R - HY$, giving

$$\boxed{e_{a,\infty} = 1 - k_3 y_\infty = 0}.$$

The derivative sensing term vanishes at a constant equilibrium. If “steady-state error” denotes the summing-junction error, the answer is zero; if it denotes actual joint-angle tracking error, it is $1 - 1/k_3$. Both vanish for a calibrated unit DC sensor, $k_3 = 1$. These final values require stability; with $\tau > 0$, sufficient and necessary second-order conditions are $1 + k_1 k_2 k_4 > 0$ and $k_1 k_2 k_3 > 0$.

4.2 P4.8: Temperature regulation and disturbance rejection

Source: textbook printed page 306–307.

Problem in brief. A unity-feedback thermal loop uses $C(s) = K/(0.2s + 1)$ and $G(s) = 100/(s^2 + 25s + 100)$. A temperature disturbance is added at the plant input. Find sensitivity to K , the disturbance transfer function, and the gain range giving an output deviation below 10% of a step disturbance.

Detailed solution. Write $D_p(s) = s^2 + 25s + 100$ and

$$Q(s) = (0.2s + 1)D_p(s) + 100K = 0.2s^3 + 6s^2 + 45s + 100(1 + K).$$

From $Y = G[C(R - Y) + T_d]$,

$$T(s) = \frac{Y}{R} = \frac{100K}{Q(s)}, \quad \boxed{S_K^T = \frac{(0.2s + 1)D_p(s)}{Q(s)}}.$$

The plant-input disturbance response is

$$\boxed{\frac{Y}{T_d} = \frac{G}{1 + CG} = \frac{100(0.2s + 1)}{Q(s)}}.$$

For $R = 0$ and $T_d = A/s$, the final output deviation is

$$y_\infty = \frac{A}{1 + K}.$$

Under the steady-state disturbance-rejection interpretation, $|y_\infty| < 0.1|A|$ requires $K > 9$ for positive K . This must be intersected with the stability range. The Routh array is

$$\begin{array}{c|cc} s^3 & 0.2 & 45 \\ s^2 & 6 & 100(1 + K) \\ s^1 & \frac{270 - 20(1 + K)}{6} & 0 \\ s^0 & 100(1 + K) & 0 \end{array}$$

All first-column entries are positive when $-1 < K < 12.5$. Therefore

$$\boxed{9 < K < 12.5 \quad \text{for stable, less-than-10\% steady-state disturbance transmission.}}$$

At $K = 9$, transmission is exactly 10%, so it is excluded by “less than.” At $K = 12.5$, the characteristic polynomial has an imaginary pair, so the steady-state calculation no longer applies.

If the requirement applies at every time: no feasible gain

The final-value result does not bound the entire transient. For example, at $K = 10$ the normalized output from a unit step disturbance peaks at approximately 0.310 before tending to $1/11$. In fact, no stable gain meets an all-time 10% bound. To prove this, normalize $A = 1$ and write the convolution equation

$$y = y_0 - K(h * y), \quad H(s) = \mathcal{L}\{h\} = \frac{G(s)}{0.2s + 1} = \frac{500}{(s + 5)^2(s + 20)}.$$

Here $h(t) \geq 0$, because H is a cascade of first-order systems with positive impulse responses. The uncontrolled plant step response and the integral of h are

$$y_0(t) = 1 - \frac{4}{3}e^{-5t} + \frac{1}{3}e^{-20t},$$

$$v(t) = \int_0^t h(u) du = 1 - \frac{20}{3}te^{-5t} - \frac{8}{9}e^{-5t} - \frac{1}{9}e^{-20t}.$$

Suppose $|y(t)| < 0.1$ at all times. Then $|(h * y)(t)| \leq 0.1v(t)$. Every stable gain has $-1 < K < 12.5$, and hence $|K| < 12.5$. Consequently, at $t = 0.1$ s,

$$y(0.1) \geq y_0(0.1) - 1.25v(0.1) = 0.236404 - 1.25(0.0414706) = 0.184566 > 0.1,$$

a contradiction. Thus the two interpretations give

$9 < K < 12.5,$ no feasible stable gain,	steady-state requirement, all-time requirement.
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The distinction is essential when stating part (c).

4.3 P4.11: Paper-stock consistency control

Source: textbook printed page 308.

Problem in brief. For the unity-feedback consistency loop with $G_c = K/(8s+1)$ and $G = 1/(3s+1)$, find the closed-loop transfer function, sensitivity to K , step steady-state error, and gain needed for 2% error.

Detailed solution. The loop transfer function is $L = K/[(8s+1)(3s+1)]$. Therefore

$$T(s) = \frac{K}{(8s+1)(3s+1) + K} = \frac{K}{24s^2 + 11s + 1 + K}.$$

The normalized sensitivity is

$$S_K^T = \frac{1}{1+L} = \frac{24s^2 + 11s + 1}{24s^2 + 11s + 1 + K}.$$

For $R = A/s$, the tracking error satisfies $E = R/(1+L)$, and hence

$$e_\infty = \frac{A}{1+K}, \quad \frac{e_\infty}{A} = \frac{1}{1+K}.$$

The second-order denominator is Hurwitz for $K > -1$, so the required positive gains satisfy the final-value assumptions. Setting the fractional error to 0.02 gives

$$\frac{1}{1+K} = 0.02 \quad \Rightarrow \quad \boxed{K = 49.}$$

Thus the least gain for an error *no greater than* 2% is 49; use $K > 49$ if a strict less-than-2% requirement is intended.

5 Chapter 5: Performance of feedback control systems

5.1 E5.10: A pole region from three step specifications

Source: textbook printed page 378.

Problem in brief. Locate the allowable second-order poles for percent overshoot at most 5%, 2% settling time below 4 s, and peak time below 1 s.

Detailed solution. Write an underdamped pole as $s = -\sigma + j\omega_d$, with $\sigma > 0$, and include its conjugate. The standard zero-free second-order overshoot relation gives

$$e^{-\pi\zeta/\sqrt{1-\zeta^2}} \leq 0.05 \quad \Rightarrow \quad \zeta \geq \frac{-\ln 0.05}{\sqrt{\pi^2 + (\ln 0.05)^2}} = 0.6901.$$

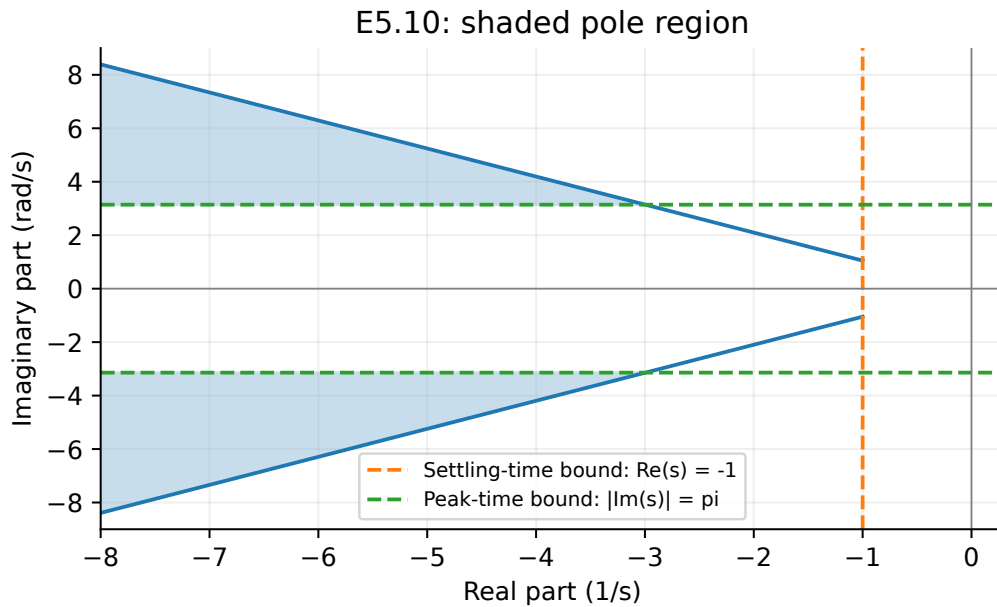
In the upper half-plane this is the wedge condition

$$0 < \omega_d \leq \sigma \frac{\sqrt{1 - 0.6901^2}}{0.6901} \approx 1.049\sigma.$$

The specified settling-time estimate gives $4/\sigma < 4$, hence $\sigma > 1$. The peak time is $T_p = \pi/\omega_d < 1$, hence $\omega_d > \pi$. The admissible region is

$$\sigma > 1, \quad \pi < |\omega_d| \leq 1.049\sigma.$$

Equivalently, it lies to the left of $\text{Re } s = -1$, inside the damping wedge about the negative real axis, and outside the horizontal strip $|\text{Im } s| \leq \pi$. The latter two requirements already imply $\sigma \gtrsim 2.996$, so the $\sigma > 1$ condition is redundant in their intersection but should still be shown.



The shaded upper and lower regions contain the admissible conjugate poles.

The settling constraint here uses the textbook approximation $T_s \approx 4/\sigma$; an exact last-band-crossing time is not identical to this line. These pole-only formulas assume the standard second-order transfer function without additional zeros.

5.2 E5.15: Exact and dominant-pole step responses

Source: textbook printed page 378–379.

Problem in brief. Plot and compare the unit-step response of

$$T(s) = \frac{2500}{(s + 50)(s^2 + 10s + 50)}$$

with the approximation obtained from its dominant complex pair.

Detailed solution. The poles are -50 and $-5 \pm 5j$. The real pole decays ten times faster than the real part of the complex pair, suggesting a good dominant-pole approximation. Preserve DC gain by replacing $s + 50$ by 50:

$$T_2(s) = \frac{50}{s^2 + 10s + 50}.$$

For a unit step,

$$Y(s) = \frac{1}{s} - \frac{1}{41(s + 50)} - \frac{10(4s + 45)}{41(s^2 + 10s + 50)}.$$

Since $s^2 + 10s + 50 = (s + 5)^2 + 25$ and $4s + 45 = 4(s + 5) + 25$,

$$y(t) = 1 - \frac{1}{41}e^{-50t} - \frac{40}{41}e^{-5t} \cos 5t - \frac{50}{41}e^{-5t} \sin 5t.$$

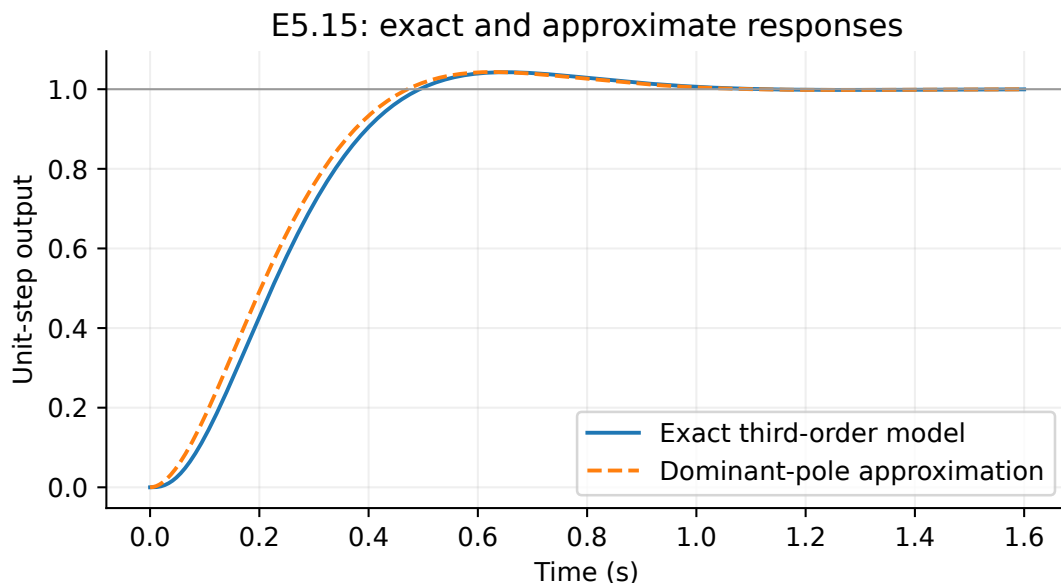
The reduced model gives

$$y_2(t) = 1 - e^{-5t}(\cos 5t + \sin 5t).$$

Both have $y(0) = 0$ and $y_\infty = 1$. The extra pole gives the exact system an additional short initial lag. Numerical peak and final-band-crossing calculations give

Quantity	Exact model	Reduced model
Percent overshoot	4.272%	4.321%
Peak time (s)	0.65045	0.62832
Actual 2% settling time (s)	0.86360	0.84324

The familiar approximation $4/5 = 0.800$ s is a settling-time estimate, not either exact crossing time above. The close curves and performance values support the dominant-pole approximation.



The fast real pole causes a small lag while leaving the main oscillatory behavior nearly unchanged.

5.3 P5.2: Dominant-pole region and a third-order design

Source: textbook printed page 380.

Problem in brief. Specify a dominant-pole region for $10\% < \text{PO} < 20\%$ and $T_s < 0.6$ s; place a third real pole sufficiently far left; and obtain a unity-feedback loop transfer function for the boundary design $T_s = 0.6$ s, $\text{PO} = 20\%$.

Detailed solution.

(a) Dominant-pole region

Invert the overshoot relation:

$$\zeta(m) = \frac{-\ln m}{\sqrt{\pi^2 + (\ln m)^2}}, \quad m = \text{PO}/100.$$

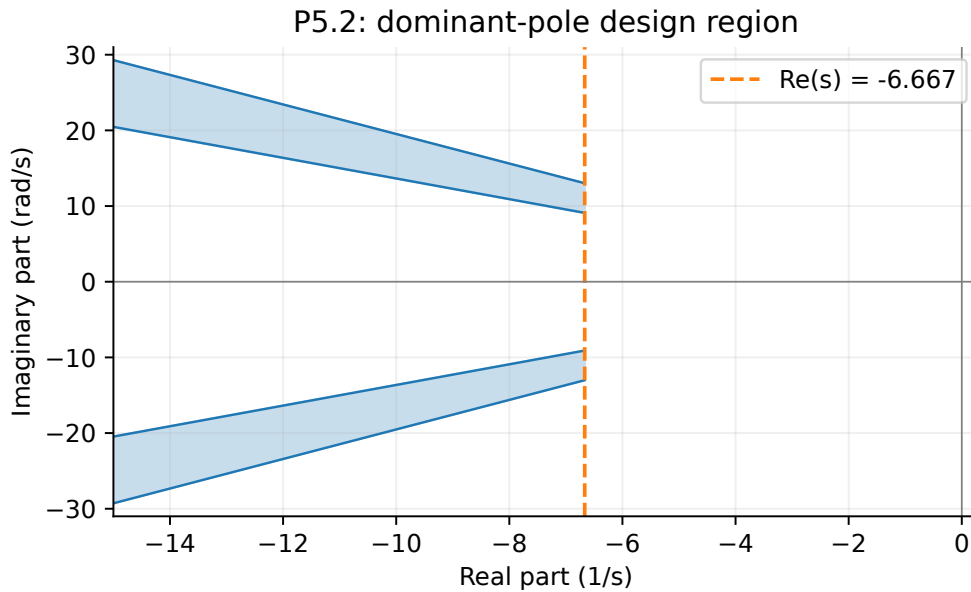
At 20% and 10%, this gives 0.45595 and 0.59116, respectively. Therefore

$$0.45595 < \zeta < 0.59116, \quad \sigma = \zeta\omega_n > \frac{4}{0.6} = 6.6667.$$

In the upper half-plane the equivalent region is

$$1.364\sigma < \omega_d < 1.952\sigma, \quad \sigma > 6.6667,$$

with a reflected region below the real axis.



The boundaries are excluded for the strict requirements of part (a).

(b) Third pole

Section 5.4 of the supplied textbook uses a ten-to-one separation of real parts for a reliable second-order approximation. Write the third root as $r_3 = -p$, $p > 0$. Then

$$p \geq 10\sigma, \quad r_3 \leq -10\sigma.$$

Near the limiting design $\sigma = 6.6667$, the minimum recommended magnitude is $p = 66.6667$. For the strict $T_s < 0.6$ region, the third-pole magnitude must exceed this limiting value and also satisfy the actual chosen 10σ bound. There is no algebraically “smallest” negative r_3 because it can be made arbitrarily far left; the meaningful smallest quantity is its required magnitude.

(c) A boundary design and its loop transfer function

Use $\sigma = 6.6667$ and $\zeta = 0.45595$. From $e^{-\pi\sigma/\omega_d} = 0.2$,

$$\omega_d = \frac{\pi\sigma}{\ln 5} = 13.0132, \quad \omega_n^2 = \sigma^2 + \omega_d^2 = 213.788.$$

Choose $p = 10\sigma = 66.6667$. A zero-free unity-DC-gain closed-loop model is

$$T(s) = \frac{p\omega_n^2}{(s+p)(s^2 + 2\sigma s + \omega_n^2)}.$$

Unity feedback implies $T = G/(1+G)$, hence $G = T/(1-T)$. The constant term cancels on subtraction, giving

$$G(s) = \frac{p\omega_n^2}{s[s^2 + (p+2\sigma)s + \omega_n^2 + 2p\sigma]} \approx \frac{14252.5}{s(s^2 + 80s + 1102.68)}.$$

More generally, any sufficiently large p gives a valid family with the same designated dominant pair. The problem's part (c) specifies boundary values even though part (a) uses strict inequalities. Also, 20% and 0.6 s are the *dominant-pair design estimates*; the actual third-order response need not attain them exactly. Exact third-order specification matching would require adjustment using the full step response, and would not uniquely fix a design without an additional choice of third pole or numerator.

For the particular unrounded choice $\sigma = 20/3$, $\omega_d = \pi\sigma/\ln 5$, and $p = 10\sigma$, independent integration of the complete third-order model gives

$$\text{PO} = 19.4665\%, \quad T_p = 0.257828 \text{ s}, \quad T_{s,2\%} = 0.584296 \text{ s}.$$

Thus this complete model actually satisfies the strict response requirements in part (a), although its designated dominant poles lie on the approximate design boundaries used in part (c).

5.4 P5.13: A second-order model of a fourth-order system

Source: textbook printed page 383.

Problem in brief. Reduce

$$G_H(s) = \frac{s^3 + 7s^2 + 24s + 24}{s^4 + 10s^3 + 35s^2 + 50s + 24} = \frac{s^3 + 7s^2 + 24s + 24}{(s+1)(s+2)(s+3)(s+4)}$$

to the second-order model displayed in the textbook, without prespecifying its poles or zero.

Detailed solution. Both models should retain DC gain 1. Write

$$G_L(s) = \frac{1 + cs}{1 + ds + es^2}.$$

Match the low-frequency Taylor coefficients through s^3 to reproduce the displayed answer.

1. Expand the original model

Normalize numerator and denominator constants to 1:

$$G_H = \frac{1 + s + \frac{7}{24}s^2 + \frac{1}{24}s^3}{1 + \frac{25}{12}s + \frac{35}{24}s^2 + \frac{5}{12}s^3 + \frac{1}{24}s^4}.$$

Long division gives

$$G_H = 1 - \frac{13}{12}s + \frac{157}{144}s^2 - \frac{1843}{1728}s^3 + O(s^4).$$

2. Expand the reduced model

Using $(1 + ds + es^2)^{-1} = 1 - ds + (d^2 - e)s^2 + (-d^3 + 2de)s^3 + \dots$,

$$G_L = 1 + (c - d)s + (d^2 - e - cd)s^2 + [-d^3 + 2de + c(d^2 - e)]s^3 + O(s^4).$$

Equating the first three nonconstant coefficients gives

$$c - d = -\frac{13}{12}, \quad d^2 - e - cd = \frac{157}{144},$$

$$-d^3 + 2de + c(d^2 - e) = -\frac{1843}{1728}.$$

The first equation gives $c = d - 13/12$. The second gives

$$e = \frac{13}{12}d - \frac{157}{144}.$$

Substitute these into the third to obtain $d = 11/8$. Thus

$$c = \frac{7}{24} = 0.2916667, \quad d = \frac{11}{8} = 1.375, \quad e = \frac{115}{288} = 0.3993056.$$

3. State and check the model

$$G_L(s) = \frac{1 + \frac{7}{24}s}{1 + \frac{11}{8}s + \frac{115}{288}s^2} \approx \frac{0.2917s + 1}{0.399s^2 + 1.375s + 1}.$$

Dividing by the exact s^2 coefficient gives numerator coefficients $84/115$ and $288/115$. The denominator poles are approximately -1.042 and -2.401 , and the zero is $-24/7 = -3.429$, reproducing the book's rounded factorization

$$G_L(s) \approx \frac{0.731(s + 3.428)}{(s + 1.043)(s + 2.4)}.$$

DC gain and the first three low-frequency moments are preserved. High-frequency behavior is also strictly proper, although the exact high-frequency coefficient is not matched.

Method clarification for the supplied edition

Section 5.8 describes matching even coefficients of $M(s)M(-s)$ and $\Delta(s)\Delta(-s)$ for the ratio $G_H/G_L = M/\Delta$. Applying *all three* conditions at s^2, s^4, s^6 literally to these data does not give the displayed model. For the normalized polynomials above, those conditions reduce to

$$144c^2 - 144d^2 + 288e + 145 = 0,$$

$$205c^2 - 60d^2 - 144e^2 + 120e + 68 = 0,$$

$$273c^2 - d^2 - 240e^2 + 2e + 29 = 0.$$

Eliminating d^2 and e^2 gives $c^2 = -737/1456 < 0$, so there is no real solution to that full set. The derivation above therefore establishes the printed target by low-frequency moment matching and explicitly records the inconsistency with a literal application of all three stated even-coefficient equations.

6 Chapter 6: Stability of linear feedback systems

6.1 E6.13: PD-controller stability region

Source: textbook printed page 432.

Problem in brief. A unity-feedback loop uses $G_c = K_P + K_D s$ and plant $4/[s(s+2)]$. Find the parameter range for stability.

Detailed solution. The characteristic equation is

$$1 + \frac{4(K_P + K_D s)}{s(s+2)} = 0 \implies q(s) = s^2 + (2 + 4K_D)s + 4K_P.$$

Its Routh array is

$$\begin{array}{c|cc} s^2 & 1 & 4K_P \\ s^1 & 2 + 4K_D & 0 \\ s^0 & 4K_P & 0 \end{array}.$$

The necessary and sufficient strict Hurwitz conditions are

$$\boxed{K_P > 0, \quad K_D > -\frac{1}{2}.$$

If controller gains are additionally required to be nonnegative, all $K_P > 0$, $K_D \geq 0$ satisfy these conditions. At $K_P = 0$ an internal root is at the origin. At $K_D = -1/2$ with $K_P > 0$, the poles are $\pm j2\sqrt{K_P}$, so the boundary is not asymptotically stable.

6.2 E6.15: A sixth-order Routh array with a zero row

Source: textbook printed page 432.

Problem in brief. Analyze

$$q(s) = s^6 + 9s^5 + 31.25s^4 + 61.25s^3 + 67.75s^2 + 14.75s + 15$$

using the Routh criterion, and determine all roots.

Detailed solution. Use exact rational coefficients to avoid numerical ambiguity. The array begins

$$\begin{array}{c|cccc} s^6 & 1 & 125/4 & 271/4 & 15 \\ s^5 & 9 & 245/4 & 59/4 & 0 \\ s^4 & 220/9 & 595/9 & 15 & 0 \\ s^3 & 406/11 & 203/22 & 0 & 0 \\ s^2 & 60 & 15 & 0 & 0 \\ s^1 & 0 & 0 & 0 & 0 \end{array}$$

For example, the s^4 first entry is $[9(125/4) - 245/4]/9 = 220/9$. The s^1 row is entirely zero, so use the auxiliary polynomial from the preceding row:

$$A(s) = 60s^2 + 15 = 15(4s^2 + 1).$$

Its roots are $s = \pm 0.5j$. Replace the zero row by the coefficients of $A'(s) = 120s$; the remaining first-column entries are 120 and 15. All first-column entries are positive, so there are no right-half-plane roots, but the auxiliary polynomial supplies an imaginary-axis pair.

Polynomial division or direct multiplication gives

$$\boxed{q(s) = (s^2 + \frac{1}{4})(s+3)(s+4)(s^2 + 2s + 5).$$

The complete root set is

$$s = -3, -4, -1 \pm 2j, \pm 0.5j.$$

The simple imaginary pair produces bounded undamped natural oscillations. Thus the system is marginally stable in the natural-response sense and is not asymptotically stable. A transfer function containing uncanceled imaginary-axis poles is not BIBO stable, so the distinction between marginal natural-response stability and input-output stability matters.

6.3 P6.5: Relative stability by shifted Routh analysis

Source: textbook printed page 434.

Problem in brief. Determine relative stability using a shifted-axis Routh test and explicit root locations for (a) $s^3 + 3s^2 + 4s + 2$, (b) $s^4 + 9s^3 + 30s^2 + 42s + 20$, and (c) $s^3 + 19s^2 + 110s + 200$.

Detailed solution. Define the stability margin $\alpha_* = -\max_i \operatorname{Re} s_i$. To test whether all roots satisfy $\operatorname{Re} s < -\alpha$, substitute $s = w - \alpha$ and apply Routh to the polynomial in w . The sign of the shift is essential.

(a) Cubic system

At $\alpha = 1$,

$$q_a(w - 1) = w^3 + w = w(w^2 + 1).$$

To perform a strict shifted Routh test just inside this boundary, set $\alpha = 1 - \varepsilon$, $\varepsilon > 0$. Then

$$q_a(w - 1 + \varepsilon) = w^3 + 3\varepsilon w^2 + (3\varepsilon^2 + 1)w + \varepsilon^3 + \varepsilon.$$

Its Routh first column is

$$1, \quad 3\varepsilon, \quad \frac{8\varepsilon^2 + 2}{3}, \quad \varepsilon^3 + \varepsilon,$$

which is positive for every $\varepsilon > 0$. At the limiting shift, roots lie on the new imaginary axis. Explicitly,

$$q_a(s) = (s + 1)(s^2 + 2s + 2), \quad \boxed{s = -1, -1 \pm j; \quad \alpha_* = 1.}$$

The complex pair and real root have the same decay rate, so a dominant-pair approximation is not justified by separation of real parts.

(b) Quartic system

At $\alpha = 1$,

$$q_b(w - 1) = w^4 + 5w^3 + 9w^2 + 5w = w(w^3 + 5w^2 + 9w + 5).$$

After explicitly extracting the zero root, the cubic quotient has Routh first column

$$1, \quad 5, \quad (45 - 5)/5 = 8, \quad 5,$$

all positive. Thus all other original roots lie strictly left of -1 , while one lies at -1 . Factoring the original polynomial gives

$$q_b(s) = (s + 1)(s + 2)(s^2 + 6s + 10),$$

$$\boxed{s = -1, -2, -3 \pm j; \quad \alpha_* = 1.}$$

The complex pair has decay rate 3, but the overall stability margin is 1 because the real pole at -1 is rightmost.

(c) Cubic system

At $\alpha = 4$,

$$q_c(w - 4) = w^3 + 7w^2 + 6w = w(w^2 + 7w + 6).$$

The quadratic quotient has Routh first column 1, 7, 6, all positive. Therefore the boundary shift is 4. The factorization confirms

$$q_c(s) = (s + 4)(s + 5)(s + 10), \quad \boxed{s = -4, -5, -10; \quad \alpha_* = 4.}$$

There are no complex roots in this case, despite the exercise's general wording about locating complex roots. All three systems are asymptotically stable. By the rightmost-root criterion, (c) has the greatest decay margin; (a) and (b) have equal overall margin, though their modal structures differ.

6.4 P6.13: Conditions on a three-parameter compensator

Source: textbook printed page 436.

Problem in brief. For unity negative feedback with $G_c = K(s+z)/(s+p)$ and $G = \omega_n^2/[s(s+2\zeta\omega_n)]$, find conditions on K, p, z for stability, assuming $K > 0$, $\zeta > 0$, and $\omega_n > 0$.

Detailed solution. Put $a = 2\zeta\omega_n > 0$ and $b = \omega_n^2 > 0$. The unreduced characteristic equation is

$$(s + p)s(s + a) + Kb(s + z) = 0,$$

or

$$q(s) = s^3 + (p + a)s^2 + (ap + Kb)s + Kbz.$$

For a monic cubic $s^3 + A_1s^2 + A_2s + A_3$, the Routh array is

$$\begin{array}{c|cc} s^3 & 1 & A_2 \\ s^2 & A_1 & A_3 \\ s^1 & (A_1A_2 - A_3)/A_1 & 0 \\ s^0 & A_3 & 0 \end{array}.$$

Strict stability therefore requires $A_1 > 0$, $A_2 > 0$, $A_3 > 0$, and $A_1A_2 > A_3$. Substitution gives

$$\boxed{p + a > 0, \quad ap + Kb > 0, \quad z > 0, \quad (p + a)(ap + Kb) > Kbz.}$$

Equivalently, for the allowed $K > 0$,

$$\boxed{p > -a, \quad K > \max\left(0, -\frac{ap}{b}\right), \quad 0 < z < \frac{(p + a)(ap + Kb)}{Kb}.}$$

If the design also assumes a stable controller pole $p > 0$, the first two conditions are automatic and the remaining useful inequality is

$$ap(p + a) + Kb(p + a - z) > 0.$$

Thus for $p > 0$ and $0 < z \leq p + a$, every $K > 0$ is stable. For $z > p + a$, stability instead requires

$$\boxed{0 < K < \frac{ap(p + a)}{b(z - p - a)}}.$$

Equality in the Routh inequality produces an imaginary-axis pair when the other coefficients are positive. These conditions retain all system modes; unstable internal modes must not be hidden by algebraic pole-zero cancellation.

7 Chapter 7: The root locus method

7.1 E7.13: Root locus with a varying pole parameter

Source: textbook printed page 520.

Problem in brief. For the unity-feedback loop $L(s) = (s + 4)/[s(s + 2)(s + z)]$, plot the closed-loop roots for $0 \leq z \leq 100$. At $z = 1, 2, 3$, estimate step overshoot and settling time from the roots, then compute the actual values.

Detailed solution.

(a) Rewrite as a root-locus problem in z

The characteristic polynomial is

$$q(s, z) = s(s + 2)(s + z) + (s + 4) = s^3 + (z + 2)s^2 + (2z + 1)s + 4.$$

Separate the varying parameter:

$$q(s, z) = \underbrace{s^3 + 2s^2 + s + 4}_{D_0(s)} + z \underbrace{s(s + 2)}_{N_0(s)}.$$

Therefore use the equivalent parameter-locus equation

$$1 + z \frac{s(s + 2)}{s^3 + 2s^2 + s + 4} = 0.$$

This avoids incorrectly treating z as a multiplicative gain in the original L . At $z = 0$, the locus starts at

$$s = -2.31460, \quad s = 0.157298 \pm 1.30515j.$$

Two branches end at 0 and -2 as $z \rightarrow \infty$; the third tends to $-\infty$. The single asymptote is at 180° , with centroid

$$\sigma_a = \frac{(-2) - (0 - 2)}{3 - 2} = 0.$$

Real-axis portions are $(-\infty, -2.31460)$ and $(-2, 0)$. The real branch from -2.31460 moves left. The conjugate pair moves into the left half-plane and meets at $s = -1$, $z = 4$, before separating along $(-2, 0)$. To check this point,

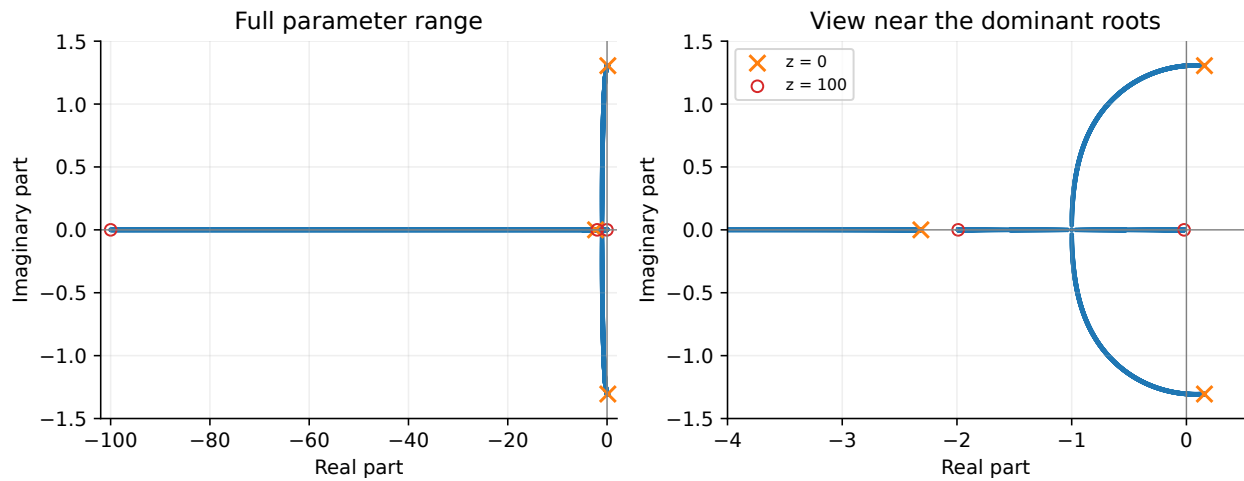
$$z(s) = -\frac{s^3 + 2s^2 + s + 4}{s(s + 2)}, \quad \frac{dz}{ds} = -\frac{(s + 1)(s^3 + 3s^2 - 8)}{s^2(s + 2)^2}.$$

The other real stationary candidate has negative z and is excluded. Indeed, $q(s, 4) = (s + 1)^2(s + 4)$.

The cubic Routh condition for $z \geq 0$ is

$$(z + 2)(2z + 1) > 4 \implies z > \frac{-5 + \sqrt{41}}{4} = 0.350781.$$

At the crossing, $\omega^2 = 2z + 1 = (\sqrt{41} - 3)/2$; the roots are $\pm 1.30444j$ and $-(z + 2)$. The plotted range stops at $z = 100$ rather than replacing that finite endpoint by an infinite-gain limit.



The left panel shows the full parameter range; the right panel resolves the dominant roots.

(b) Dominant-pair estimates

At the requested values, the roots and estimates are

z	Real root	Complex pair	Estimated PO	Estimated T_s (s)
1	-2.44225	$-0.278875 \pm 1.24902j$	49.59%	14.343
2	-2.69562	$-0.652190 \pm 1.02885j$	13.65%	6.133
3	-3.20557	$-0.897215 \pm 0.665457j$	1.447%	4.458

For example, at $z = 1$, $\sigma = 0.278875$ and $\omega_d = 1.24902$, so

$$PO_{\text{est}} = 100e^{-\pi(0.278875)/1.24902} = 49.59\%, \quad T_{s,\text{est}} = 4/0.278875 = 14.343 \text{ s}.$$

These are pole-based second-order estimates, not exact third-order response properties. The closed-loop zero at -4 and finite real pole change the modal amplitudes.

(c) Actual response and numerical performance

For each z ,

$$Y_z(s) = \frac{s+4}{s[s^3 + (z+2)s^2 + (2z+1)s + 4]}.$$

All three requested systems have $y_\infty = 1$. Let the real pole be $-r$ and the complex pair be $-\sigma \pm j\omega_d$. A real inverse-transform expression is

$$y_z(t) = 1 + ae^{-rt} + e^{-\sigma t}[b \cos(\omega_d t) + c \sin(\omega_d t)],$$

with the following coefficients:

z	r	σ	ω_d	a	b	c
1	2.442250	0.278875	1.249025	-0.102213	-0.897787	-0.400312
2	2.695621	0.652190	1.028852	-0.092448	-0.907552	-0.817514
3	3.205569	0.897215	0.665457	-0.042941	-0.957059	-1.497225

These coefficients come from residues. For a characteristic root λ_i ,

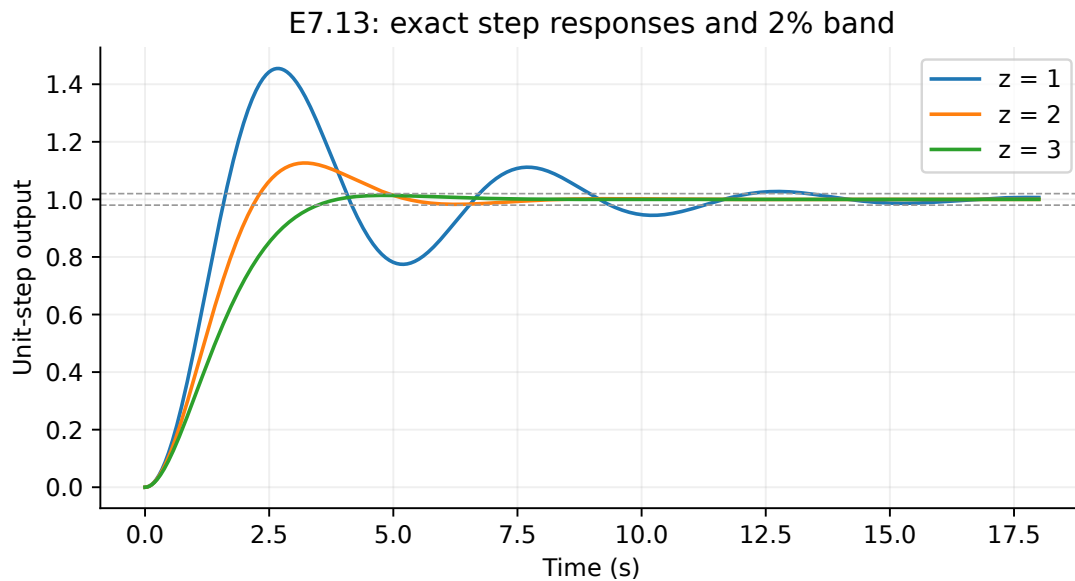
$$\text{Res}_{s=\lambda_i} Y_z(s) = \frac{\lambda_i + 4}{\lambda_i q'(\lambda_i)}.$$

A conjugate residue $u + jv$ contributes $2e^{-\sigma t}[u \cos \omega_d t - v \sin \omega_d t]$. The pole at zero has residue 1.

To determine the exact response metrics numerically, solve $\dot{y}_z(t) = 0$ for the positive stationary times, select the largest output peak, and compute $100[\max y_z(t) - 1]$. Solve $y_z(t) = 0.98$ and $y_z(t) = 1.02$ and select the last crossing for settling time. This gives

z	Actual PO	Actual peak time (s)	Actual 2% T_s (s)
1	45.482%	2.67565	13.36339
2	12.655%	3.21728	4.89449
3	1.3949%	4.82525	3.47857

For $z = 3$, the peak is below 1.02; therefore settling occurs before the peak, which is entirely consistent with the 2% band definition. The estimates overpredict both overshoot and settling time here, with the settling approximation especially conservative for $z = 2, 3$.



Exact unit-step responses. The dashed horizontal lines mark the 2% band.

7.2 E7.26: A parameter-dependent state matrix

Source: textbook printed page 521.

Problem in brief. Given

$$A = \begin{bmatrix} 0 & 1 \\ 3-K & -2-K \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [1 \quad -1],$$

find the characteristic polynomial, sketch the roots for $K \geq 0$, and determine stability.

Detailed solution. The characteristic determinant is

$$q(s, K) = \det \begin{bmatrix} s & -1 \\ K-3 & s+2+K \end{bmatrix} = s^2 + (K+2)s + K-3.$$

To express it as a conventional positive-parameter root locus,

$$q = (s-1)(s+3) + K(s+1), \quad \boxed{1 + K \frac{s+1}{(s-1)(s+3)} = 0.}$$

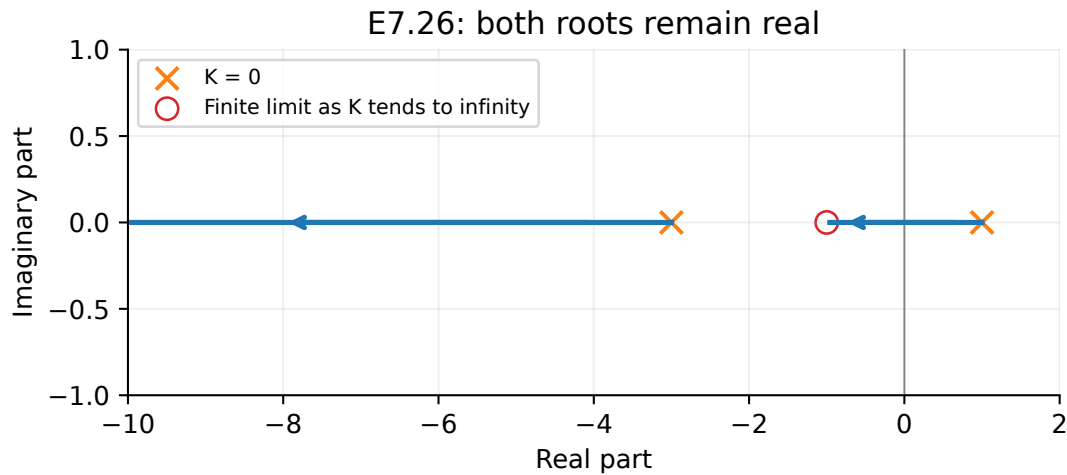
The starting poles at $K = 0$ are 1 and -3 , and the finite zero is -1 . The real-axis segments are $(-\infty, -3)$ and $(-1, 1)$. The one asymptote has angle 180° and centroid

$$\sigma_a = (1-3) - (-1) = -1.$$

The exact roots are

$$s_{\pm}(K) = -\frac{K+2}{2} \pm \frac{\sqrt{K^2+16}}{2}.$$

Their discriminant $K^2 + 16$ is always positive, so both branches remain on the real axis and there is no breakaway. As K grows, the root at 1 moves left toward -1 and the root at -3 moves left toward $-\infty$.



The arrows show increasing K ; the finite limiting root is -1 .

The second-order Hurwitz conditions are $K + 2 > 0$ and $K - 3 > 0$. With $K \geq 0$,

$$K > 3 \quad \text{is necessary and sufficient for asymptotic stability.}$$

At $K = 3$ the roots are $0, -5$; for $0 \leq K < 3$ one root is positive. Matrix C changes the measured output but does not change the eigenvalues or the characteristic polynomial of A . In fact,

$$G(s) = C(sI - A)^{-1}B = \frac{1-s}{s^2 + (K+2)s + K-3},$$

which is a useful independent check. The parameter-locus equation above describes eigenvalue motion; it does not assume an additional external unity-feedback loop around this state model.